Growth Stocks and the Petersburg Paradox by David Durand

(Source: *The Journal of Finance*, September 1957)

The allure of growth stocks is partly in the excitement that is naturally stimulated by highly successful ever booming businesses—and partly in the potential for very large profits to investors. Ah, but what price should you pay? Author, Durand, linked this question to the Petersburg Paradox, a problem in valuation presented by Daniel Bernoulli.

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This article will explain why using **high PERPETUAL growth rates** in a discounted cash flow formula result in nonsense. See here: http://thismatter.com/money/stocks/valuation/dividend-discount-model.htm

 This is a classic investment article referred to by Benjamin Graham in his chapter, *Newer Methods for Valuing Growth Stocks in 4th Edition of Security Analysis (1962)*. (Stay tuned for that chapter to be posted).

*Graham: "It is important for the student to understand why this pleasingly simple method of valuing a common stock or group of stocks had to be replaced by more complicated methods, especially in the growth-stock field. It would work fairly plausibly for assumed growth rates up to, say, 5 percent. The latter figure produces a required dividend return of only 2 percent, or a multiplier of 33 for current earnings, if payout is two-thirds. **But when the expected growth rate is set progressively higher, the resultant valuation of dividends or earnings increases very rapidly.** A 6.5% growth rate produces a multiplier of 200 for the dividend, and a growth rate of 7 percent or more makes the issue worthy of infinity if it pays any dividend. In other words, on the basis of this theory and method, no price would be too much to pay for such a common stock.[1]*

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At a time like the present, when investors are avidly seeking opportunities for appreciation, it is appropriate to consider the **difficulties of appraising growth stocks.** There is little doubt that when other things are equal the forward-looking investor will prefer stocks with growth potential to those without. **But other things rarely are equal—particularly in a sophisticated market that is extremely sensitive to growth.** When the growth potential of a stock becomes widely recognized, its price is expected to react favorably and to advance far ahead of stocks lacking growth appeal, so that its price-earnings ratio and dividend yield fall out of line according to conventional standards. Then the choice between growth and lack of growth is no longer obvious, and the **astute investor must ask whether the market price correctly *discounts* the growth potential. It is possible that the market may, at times, pay too much for growth?**

Most problems encountered in appraising growth stocks seem to fall into two categories. First there are the practical difficulties of *forecasting* sales, earnings, and dividends. Then comes the theoretical

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[1] David Durand has commented on the parallel between this aspect of growth-stock valuation and the famous mathematical anomaly known as the "Petersburg Paradox."

difficulties of reducing these forecasts to present values. **For a long time it seems to have been assumed, altogether too casually, that the present value of a forecasted dividend stream could be represented simply as the sum of all expected future payments discounted at a uniform rate.** Doubts, however, are beginning to manifest themselves. As early as 1938, J. B. Williams suggested non-uniform discount rates, varying from payment to payment.[2] More recently, Clendenin and Van Cleave have shown that discounting forecasted dividends at a uniform rate in perpetuity may lead to absurdities or paradoxes, since implied present value of infinity sometime result. "We have not yet seen any growth stocks marketed at the price of infinity dollars per share," they remark, "but we shall hereafter be watching. Of course, many investors are skeptical and would probably wish to discount the large and remote dividends in this perpetually growing series at a high discount rate, thus reducing our computed value per share to a figure somewhat below the intriguing value of infinity."[3] Clendenin and Van Cleave might have made a good point even better had they noticed a remarkable analogy between the appraisal of growth stocks and the famous Petersburg Paradox, which commanded the attention of most of the important writers on probability during the eighteenth and nineteenth centuries.

THE PETERBURG PARADOX

In 1738 Daniel Bernoulli presented before the Imperial Academy of Sciences in Petersburg a classic paper on probability, in which he discussed the following problem attributed to his cousin Nicholoas: "Peter tosses a coin and continues to do so until it should land 'heads' on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation."[4]

Sequence of Tosses	Probability	Payment
H	½	1
TH	¼	2
TTH	1/8	4
TTTH	1/16	8
TTTTH	1/32	16

One may easily obtain a solution according to the principles of mathematical expectation by noting the sequence of payments and probabilities in Figure1: Paul's expectation is the sum of the products of probability by payment or ½ + 2/4 + 4/8 + 8/16 + 16/32 + …..

If the players agree to terminate the game of n tosses, whether a head shows or not, the series will contain n terms and its sum will be n/2; but if they agree to continue without fail until a head shows, as the rules of the game stipulate, then n is infinite and the sum n/2 is infinite as well. **Thus the principles of mathematical expectation imply that Paul should pay an infinite price to enter this game, but this is a conclusion that virtually no one will accept.** A variety of explanations have been given to show that

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[2] *John B. Williams, The Theory of Investment Value, 1938), pages 50-60.*
[3] John C. Clendenin and Maurice Van Cleave, "Growth and Common Stock Values," Journal of Finance 9 (1954), 23-36.
[4] *Daniel Bernoulli, "Exposition of a New Theory on the Measurement of Risk," Econometrica 22 (1954), 23-36*

the value of the game to Paul is, in fact, only a finite amount—usually a small finite amount; and all of the explanations are relevant to growth stock appraisal….

ATTEMPTS TO RESOLVE THE PETERSBURG PARADOX[5]

The many attempts to resolve the paradox, summarized very briefly below, fall mostly into two broad groups: those denying the basic assumptions of the game as unrealistic, and those arguing from additional assumptions that the value of the game to Paul is less than its mathematical expectation.

The basic assumptions of the game are open to all sorts of objections from the practically minded. How, in real life, can the game continue indefinitely? For example, Peter and Paul are mortal; so, after a misspent youth, a dissipated middle age, and a dissolute dotage, one of them will dies, and the game will cease—heads or no heads. Or again, Peter's solvency is open to question, for the stakes advance at an alarming rate. With an initial payment of one dollar, **Peter's liability after only 35 tails exceeds the gold reserve in Fort Knox**, and after only three more, it exceeds the volume of bank deposits in the United States and approximately equals the national debt.

With this progression, the sky is, quite literally, the limit. Even if Peter and Paul agree to cease after 100 tosses, the stakes, though finite, stagger the imagination.

Despite these serious practical objections, a number of writers chose to accept the assumption of an indefinitely prolonged game at face value, and to direct their attention toward ascertaining the value of such a game to Paul. First among these was the Swiss mathematician Gabriel Cramer, who early in the eighteenth century proposed two arbitrary devices for resolving the Petersburg Paradox by assuming that the utility of money is less than proportional to the amount held. First, if the utility of money is less than proportional to the amount held. First, if the utility of money is proportional to the amount up to $2^{24} = 166,777,216$ ducats and constant for amounts exceeding 2^{24}, so that the utility of the payments ceases to increase after the 24th toss, Paul's so-called moral expectation is about 13 ducats. Second, if the utility of money is assumed equal to the square root of the amount held, Paul's moral expectation is only about 2.9 ducats. Cramer believed that 2.9 was a more reasonable entrance fee than 13.

A little later and apparently independently, Daniel Bernoulli devised a solution only slightly different from Cramer's **assuming that the marginal utility of money is inversely proportional to the amount held;** he derived a formula that evaluates Paul's expectation in terms of his resources at the beginning of the game. From this formula, which does not lend itself to lightning computation, Bernoulli estimated roughly that the expectation is worth about 3 ducats to Paul when his resources are 10 ducats, about 4 ducats when his resources are 100, and about 6 when his resources are 1000, At this rate, Paul must have infinite resources before he can value his expectation at infinity; but then, even his infinite valuation will constitute only an infinitesimally small fraction of his resources.

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[5] For a general history of the paradox, see Isaac Todhunter, a *History of the Mathematical Theory of Probability from the Time of Pascal to that of Lapace*

An interesting variant of Bernoulli's approach was proposed about a century later by W.A. Whitworth[6]-- at least, some of us would consider it a variant though its author considered it an entirely different argument. Whitworth was, in fact, seeking a solution to the Petersburg problem that would be free of arbitrary assumptions concerning the utility of money; and he derived a solution by considering the risk of gamblers' ruin, which is always present when players have limited resources. Thus, for example, if A with one dollar matches pennies indefinitely against B with $10, it is virtually certain that one of them will eventually be cleaned out; furthermore, A has 10 chances out of 11 of being the victim. Accordingly, a prudent A might demand some concession in the odds as the price of playing against B But how much concession? **Whitworth attacked this and other problems by assuming a prudent gambler will risk a constant proportion of his resources, rather than a constant amount, on each venture; and he devised a system for evaluation ventures that entail risk of ruin.** Applied to the Petersburg game, this system indicates that Paul's entrance fee should depend upon his resources. Thus Whitworth's solution is reminiscent of Bernoulli's –particularly when one realizes that Whitworth's basic assumption implies an equivalences between a dime bet for A with $1 and a dollar bet for B with $10. Bernoulli, of course, would have argued that the utility of a dime to A was equal to the utility of a dollar to B. Finally, the notion of a prudent gambler seeking to avoid ruin has strong utilitarian undertones**; for it implies that the marginal utility of money is high when resources are running out.**

But Whitworth's approach—regardless of its utilitarian subtleties—is interesting because it emphasizes the **need for diversification**. **The evaluation of a hazardous venture—be it dice game, business promotion, or risky security---depends not only on the inherent odds, but also on the proportion of the risk-taker's resources that must be committed.** And just as the prudent gambler may demand odds stacked in his favor as the price for betting more than an infinitesimal proportion of his resources, so may the prudent portfolio manager demand a greater than normal rate of return (after allowing for the inherent probability of default) as the price of investing more than an infinitesimal proportion of his assets in a risky issue….

Although the preceding historical account of the Petersburg Paradox has been of the sketchiest, it should serve to illustrate an important point. **The various proposed solutions, of which there are many, all involve changing the problem in one way of another.** Thus some proposals evaluate the cash value of a finite game, even when the problem specifies an infinite game; others evaluate the utility receipts, instead of the cash receipts, of an infinite game; and still others forsake evaluation for gamesmanship and consider what Paul as a prudent man should pay to enter. But although none of these proposals satisfy the theoretical requirements of the problem, they all help to explain why a real live Paul might be loath to pay highly for his infinite mathematical expectation. As Keynes aptly summed it up, "We are unwilling to be Paul, partly because we do not believe Peter will pay us if we have good fortune in the tossing, partly because we do not know what we should do with so much money….if we won it, partly because we do not believe we should ever win it, and partly because3 we do not think it would be an rational act to risk an infinite sum or even a very large sum for an infinitely larger one, whose attainment is infinitely unlikely."

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[6] W. A. Whitworth, *Choice and Change, 4th Edition, enlarged, 1886*

IMPLICATIONS OF PETERSBURG SOLUTIONS FOR GROWTH-STOCK APPRAISAL

If instead of tossing coins, Peter organizes a corporation in a growth industry and offers Paul stock, the latter might be deterred from paying the full discounted value by any of the considerations that would deter him from paying the full mathematical expectation to enter the Petersburg game. And again, these considerations fall into two categories: first, those denying the basic assumptions concerning the rate of indefinitely prolonged growth; and second, those arguing that the value of the stock to Paul is less than its theoretical discounted value.

Underlying J.B. Williams'….. (way at looking at the problem is) the assumption that Peter, Inc., will pay dividends at an increasing rat g for the rest of time….A slightly different assumption…is that Peter will pay steadily increasing dividends until the game terminates with the toss of a head, and that the probability of a head will remain forever constant a $i/(1 + i)$. **Under neither assumption is there any provision for the rate of growth ever to cease or even decline. But astronomers now predict the end of the world within a finite number of years—somewhere in the order of 10,000,000,000**—and realistic security analysts may question Peter, Inc., ability to maintain a steadily increasing dividend rate for anywhere near that long. Williams, in fact regarded indefinitely increasing dividends as strictly hypothetical, and he worked up formulas for evaluating growth stocks on the assumption that dividends will follow a growth curve (called a logistic by Williams) that increases exponentially for a time and then levels off to an asymptote. This device guarantees that the present value of any dividend stream will be finite, no matter how high the current, and temporary rate of growth. Clendenin and Van Cleave, though not insisting on a definite ceiling, argued that continued rapid growth is possible only under long-run price inflation.

The assumption of indefinitely increasing dividends is most obviously objectionable when the growth rate equals or exceeds the discount rate ($g >$ or $=$ to i) and the growth series… sums to infinity….If Peter, Inc. is to pay a dividend that increases at a constant rate $g >= I$ per year, it is absolutely necessary, though not sufficient, that he earn a rate on capital, $r = E/B$, that is greater than the rate of discount—more exactly, $r >= i/(1-p)$. **But this situation poses an anomaly, at least for the equilibrium theorist, who argues that the marginal rate of return on capital must equal the rate of interest in the long run.** How, then, can Peter, Inc. continually pour increasing quantities of capital into his business and continue to earn on these accretions a rate higher than the standard rate of discount? This argument points toward the conclusion that growth stocks characterize business situations in which limited, meaning finite though not necessarily small, amounts of capital can be invested at rates higher than the equilibrium rate. If this is so, then the primary problem of the growth-stock appraiser is to estimate how long the departure from equilibrium will continue, perhaps by some device like Williams' growth curve.

If, for the sake of argument, Paul wishes to assume that dividend growth will continue infinitely at a constant rate, he can still find reason s for evaluating Peter's stock at somewhat less than its theoretical value just as he found reasons for evaluating his chances in the Petersburg game at less than the mathematical expectation. The decreasing –marginal-utility approach of Cramer and Bernoulli implies that the present utility value of a growing dividend stream is less than the discounted monetary value, because the monetary value of the large dividends expected in the remote future must be substantially

scaled-down in making a utility appraisal. Or again, Whitworth's diversification approach implies that a prudent Paul with finite resources can invest only a fraction of his portfolio in Peter's stock; otherwise he risks ruinous loss. And either argument is sufficient to deter Paul from offering an infinite price, unless, of course, his resources should be infinite.

The moral of all this is that conventional discount formulas do not provide completely reliable evaluations. Presumably they provide very satisfactory approximations for high-grade, short-term bonds and notes. But as quality deteriorates or duration lengthens, the approximations become rougher and rougher. **With growth stocks, the uncritical use of conventional discount formulas is particularly likely to be hazardous; for, as we have seen, growth stocks represent the ultimate in investments of long duration.** Likewise, they seem to represent the ultimate in difficulty of evaluation. The very fact that the Petersburg problem has not yielded a unique and generally acceptable solution to more than 200 years of attack by some of the world's great intellects suggest, **indeed, that the growth-stock problem offers no great hope of a satisfactory solution.**