Response: Remain Steadfast With the St. Petersburg Paradox to Quantify Irrational Exuberance

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In reading Professor Belden's comments we find that the author has indirectly aimed those comments toward the article by Durand (1957) which is, in part, the basis for our article. In framing our response we therefore begin with biographical details about Durand's professional achievements. Subsequently, we turn to a response to the comments in question.

We also received from Professor H. C. Tijms (Department of Econometrics and Operations Research, Free University, Amsterdam, The Netherlands) some remarks on the comments we made in our article regarding results of Whitworth (1901). We will provide further details of Whitworth's results, including their historical context.

1. SOME REMARKS ON THE WORK OF DAVID DURAND

In preparing these biographical remarks about Durand, we searched the Internet in great detail using a variety of search engines; for example, Google, Yahoo, Metacrawler. Although Internet searches are never perfect, the material we located is extensive. We believe that it conveys a true picture of the great breadth and high quality of Durand's professional accomplishments.

We quote liberally from the memorial to Durand provided by his friend and colleague, Enders Robinson (1996); from the March 6, 1996, edition of *Tech Talk*, the MIT official newspaper, and other sources. In almost all of what follows, we quote or paraphrase those sources.

Robinson (1996) wrote, "David Durand, professor emeritus of management at the Massachusetts Institute of Technology and a pioneer in the application of statistical tools to problems in corporate finance and other fields, died on February 26, 1996, at the age of 83. Professor Durand was reared in Ithaca, New York, and received his B.A. degree from Cornell University in 1934, his master's degree (1938), and doctorate (1941) from Columbia University. He was a lieutenant in the U.S. Naval Reserve during World War II, serving in the Hawaiian Islands and on Guam. Before joining MIT in 1953, Durand was associated with the National Bureau of Economic Research, then in Riverdale, NY, and the Institute of Advanced Study in Princeton, NJ. . . . David's winning personality immediately attracted Albert Einstein and a friendship developed. I never tired of listening to David's stories about the great master."

Robinson also noted that "Professor Durand's first appointment at MIT was as a research associate at the Sloan School of Management. He became an associate professor in 1955, professor in 1958, and he retired in 1973. . . . [Durand] from the outset formed a close association with Professor George Wadsworth and me. David Durand was a foremost expert on time series analysis and mathematical modeling. He provided valuable insight that led to the development of time-series models for geophysical data analysis. David was an early pioneer in the use of computers, and from 1953 to 1956 he wrote some of the basic computer programs still used today at seismic processing centers worldwide. He was always generous with his time. His devotion and sense of humor made the hours seem to fly by as we worked on the MIT computer. My association with David has been continuous ever since."

Robinson continued, "Each time we met, [Durand] would come up with novel ideas that could always be transformed into valuable new methods of time series analysis. David was for many years an associate editor of *Financial Management* and he wrote the textbook *Stable Chaos* (Durand 1971) which I always keep handy by my desk. David is best known for his development of the statistical explanation for the level of yields on long-term and short-term bonds. This fundamental work forms the cornerstone of every book on financial analysis. Geophysicists will remember David each time they enter digital seismic data into a computer for, of all the MIT professors associated with the Mathematics Department seismic project in the 1950s, he was the only one who knew how to use a digital computer and spent many long hours with me to get seismic programs to work."

From MIT's *Tech Talk*, we learned that "In addition to the application of statistical methods to financial problems, Durand's fields of specialization included term structure of interest rates and statistics. His research in finance included a sampling analysis of default experience for consumer installment loans, farm mortgage lending experience, and factors affecting bank stock prices. Durand's work with statistical methodology and techniques involved the early use of punch-card equipment for general statistical tabulation as well as for mathematical computation." *Tech Talk* noted that Durand authored "many articles for professional journals. He also taught part-time at Columbia University and did consulting work for the Twentieth Century Fund."

Tech Talk continued, "Some of Dr. Durand's strongly held views stirred lively debate with other members of the management faculty. One of his former doctoral students, Don Lewin, said that Dr. Durand 'used his keen intellect and statistical knowledge and skills to develop many ideas' and to question whether statistical models matched reality. 'Frequently, this did not endear him to those enamored of a model. Indeed, his doubting approach caused him to be often in the center of a controversy.' " Dr. Durand also insisted, Dr. Lewin said, that the model builder rely heavily on his or her own judgment. In *Stable Chaos*, Dr. Durand wrote, "Systematic procedures and objective tests serve to strengthen the analyst's judgment, not to replace it; they

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enable him to learn more quickly and more effectively from his own experience, and to sharpen his critical faculties."

"Dr. Durand also championed good writing and enlivened some of his own journal articles with intriguing figures of speech. In one, he wrote: 'To suppose that any imaginative analyst or responsible financial manager, interested in a comprehensive view, would be content to base an important appraisal and the subsequent investment decision on just one of the many useful numbers available is on par with supposing that a hungry gourmet at a smorgasbord would be content to make a whole meal of pickled herring . . . ' "

"Another former student, Dr. Paul D. Berger, professor and department chair in Quantitative Methods and Marketing at the Boston University School of Management, recalled Dr. Durand as 'a special teacher and mentor to many students. He had a 'jolly' manner about himself that set students at ease and allowed them to enjoy the material he imparted to them. He cared about people and was dedicated to academic integrity and excellence.' " As regards Professor's Durand's "winning personality," "sense of humor," and "jolly manner" we strongly recommend to readers the letter by Durand (Beelzebub 1970) which appears in a *Dictionary for Statismagicians*. Then and now, there is an enduring nature to the amusing definitions of terms which arise in everyday statistical practice.

Paulo (2003, p. 330) noted, "Only in 1952 did David Durand of MIT propose the then unorthodox position that the financial goal of a business should be to maximize the investment value of the firm rather than to maximize income (Durand 1952, pp. 215–247)." Paulo goes on to make clear the importance of Durand's proposition for subsequent research on the financial value of a firm.

From three Nobel laureates came acknowledgments to Durand and his effect on the thinking of economists. In the March 6, 1996, edition of *Tech Talk*, Paul Samuelson noted that during Durand's tenure at the National Bureau of Economic Research, he [Durand] "pioneered the empirical study of how long-term bonds usually require a higher yield than short. Everyone understands that today, but he was the first to document it." Modigliani and Miller (1958) mentioned Durand in their thinking toward the formulation of their famous theorem. In an interview by Barnett and Solow (2000), Modigliani commented (p. 223) that on "listening to a paper by David Durand suggesting (and then rejecting) the so-called 'entity theory' of valuation, I gradually became convinced of the hypothesis that market value should be independent of the structure of financing . . . This result later became part of the proof of the Modigliani-Miller theorem." Many years later, Durand (1989) provided his "afterthoughts" on the controversy with Modigliani and Miller; his "new thoughts on growth and the cost of capital" indicate that, even after retirement, he remained active in research.

We searched the journal repository JSTOR (http://www.jstor.org) for Professor Durand's publications and references to his work. A review of his publications reveals an astonishing breadth of articles authored by him, with a commensurately large number of references to his work. His publications reveal that he would have been professionally at home in a department of statistics,

economics, geophysics, finance, or mathematics. The man truly was a polymath.

In summary and in repetition, we find that David Durand was a patriot, that he cared about people, was dedicated to academic integrity and excellence, was always generous with his time, produced fundamental work in every area in which he developed an interest, performed substantial administrative service to the American Statistical Association and to the statistical profession, had a sense of humor and a jolly manner which set students at ease, and that he championed good writing. To the extent that we hold a personal brief for him, it is because we have learned much from reading his papers.

We never met David Durand but we sincerely wish that we had done so.

2.   RESPONSE TO PROFESSOR BELDEN

The introduction to Professor Belden's letter states "that the parameters in the growth stock valuation formula do not parallel those in the St. Petersburg Paradox." We do not comprehend this statement, because "parameters"—as used in a mathematical or a quotidian sense—cannot "parallel" anything. In the next sentence, we read that "stock valuation is more complex than this statistical paradox suggests" so we infer that she views the St. Petersburg paradox as inapplicable to the valuation of growth stocks. Judging from the reviews and many citations of Durand's article, the St. Petersburg paradox indeed has a clear connection with the valuation of growth stocks. Moreover, Professor Belden's views are countered by financial analysts (e.g., Mauboussin and Bartholdson 2003) who have noted for the benefit of portfolio managers the significance of Durand's article, and from many other references which can be located by a simple Internet search using the key words "*Durand Petersburg paradox*." Professor Belden should keep in mind that the paradox is an abstract statement, not a growth-stock valuation formula.

We think that Professor Belden has misunderstood both Durand (1957) and Székely and Richards (2004), for neither article overlooked the importance of speculator psychology. Indeed, both articles used the St. Petersburg paradox to *quantify* the extent to which speculator psychology has attained an irrational bent. We sense that Professor Belden is surprised by the simplicity and ease with which the St. Petersburg paradox could, and *should*, have been used by mutual-fund analysts to assess the irrationality of stock prices in the late 1990s.

When Alan Greenspan posed his now-famous question, "But how do we know when irrational exuberance has unduly escalated asset values . . .?," we gave serious thought to his words—if only because our own pension funds were sure to be affected by the ensuing debate. Although Greenspan's question gained prominence for the phrase "irrational exuberance," we believed that the crucial phrase was "But how do we *know*" [our emphasis]. After reviewing classic texts on the analysis of financial securities (e.g., Graham 1985), we concluded that stock prices were irrational. We also felt it incumbent upon us, as statisticians, to *quantify* the extent to which speculator irrationality had progressed. On analyzing Durand's remarkable article, we knew that it was time to retreat to the relative safety of fixed-income assets. Until early 2000, we then followed the instructions of Clendenin and Van Cleave (1954), watching carefully—and with bemusement—for "growth stocks marketed at the price of infin-

ity dollars per share." Luckily for growth stock buyers, we were unsuccessful in our watchful efforts.

The preceding comments reinforce the importance of the St. Petersburg paradox as *one of many* tools for assessing speculator irrationality. It is correct, as noted by Professor Belden, that "investors are not always rational calculators." This is precisely why *quantitative* early-warning systems are needed to avoid financial tsunamis. In our view, the St. Petersburg paradox is superb in this regard.

Whether or not the term "behavioral finance" existed before 1957 is immaterial to the relationship between the St. Petersburg paradox and the valuation of growth stocks. Long before 1957, quantitative scholars were well aware of the difficulty of quantifying irrationality in speculator behavior. Isaac Newton, upon losing 20,000 pounds in the South Sea Bubble (which sold nothing but plans and ideas and lost 85% of its value in the fall of 1720) said, "I can calculate the motions of heavenly bodies, but not the madness of people." The classic book of MacKay (1852) surely can be seen as an earlier entry in the field of behavioral finance, for it chronicles the development of speculator irrationality in much the way that medical texts review the development of a human disease. We refer also to Graham and Dodd (1940), Kindleberger (1989), and Galbraith (1993) for extensive discussions of quotidian "behavioral finance" existing long before the term was given an academic definition.

We are at a loss as to what to make of Professor Belden's statement that "the outrageous prices for growth stocks in 1999 had very little to do with mathematics." On the one hand, those "outrageous prices" obviously had many causes; we offer The Seven Deadly Sins as a partial list, and Shiller (2000) offered additional causes too. Nevertheless, a Google search using the key words "*1999 growth stock valuation dividend discount model*" returns a lengthy list of references which demonstrate that a widespread cause of investor irrationality was the outrageous valuations provided by securities analysts working with dividend-discount models. On the other hand, and this is a point made by Székely and Richards (2004), had speculators performed the *elementary* mathematical calculations embodied by Durand's reformulation of the St. Petersburg paradox or those offered by Graham (1985), few speculators would have spent the past five years repeating those two, most important words, "If only."

Professor Belden claims that "stock valuation . . . can be better understood using the principles of behavioral finance that describe investor psychology." We beg to differ. We believe that in art, music, love, real estate, fine wines, *haute cuisine*, and all human activities, one should have firm convictions about what works best for the individual. Perhaps behavioral finance is better for some, but it is not for us. As investors, we cling mightily to the old-fashioned valuation methods of Graham and Dodd (1940), Graham (1985), and Whitman and Shubik (1979). We note that these old-fashioned methods saved many an investor from the terrible losses of 2000, whereas Professor Belden's account of behavioral finance suggests that it did little more than provide an after-the-fact explanation of why such losses took place. Even if behavioral finance were better, we cannot resist the temptation to imitate Professor Durand by writing, "To suppose that any imaginative analyst or responsible financial manager, interested in a comprehensive view, would be content to base an important

appraisal and the subsequent investment decision on [behavioral finance only] is on par with supposing that a hungry gourmet at a smorgasbord would be content to make a whole meal of pickled herring."

With the comment that behavioral finance may be applied to "describe" investor psychology, Professor Belden misses the point of Durand's paper entirely. From the very outset, Durand makes it clear that what is needed is a *quantitative* assessment of investor irrationality. To illuminate the difference between Professor Belden's approach and ours, consider the hypothetical case of Joe Neophyte, a novice speculator. Neophyte has developed a burgeoning interest in purchasing for the long term ten shares of common stock in the growth company Google, the Internet search engine and advertiser. On August 19, 2004, Google's common stock opened trading with great fanfare at an initial price of $100 per share. At time of writing (February 16, 2005), Google's stock has increased to over $198 per share. There has been some discussion recently of whether Google's current stock price is justified, so we infer that there is evidence for and against the hypothesis that Google's stock buyers have experienced "irrational exuberance." Neophyte, a cautious guy, is sure to ask, "But how do I *know* when irrational exuberance has unduly escalated the value of Google's stock?" Professor Belden would point Neophyte to behavioral finance, but her letter indicates that he will find there only a *qualitative* assessment of existing exuberance.

For a *quantitative* measurement of the extent to which Google's common stock may have experienced irrational exuberance since its initial public offering, we would recommend to Neophyte the virtues of Durand's reformulation of the St. Petersburg paradox within the context of growth stock valuation: Compare financial analysts' estimates of g, the annual compound growth rate of Google's revenue per share over the *long term*, with i, a measure of interest rates as determined by the Federal Reserve's discount rate or the current rate on ten-year U.S. Treasury bonds. If $g \geq i$, then we think Neophyte would be wise to bypass Google's stock. As the reader observes, Neophyte has derived *via* the St. Petersburg paradox a quantitative assessment of any existing exuberance.

Let us provide a real-time implementation of our advice to Neophyte. Turning to the Web site http://finance.yahoo.com/q/ae?s=GOOG we find that 19 financial analysts currently estimate an average one-year growth-rate of 35.6% for Google's revenue in 2006. If it were proposed to Neophyte that Google's current stock price can be justified by means of a dividend-discount model using a long-term growth rate of 35.6% then, by applying Durand's reformulation of the St. Petersburg paradox, Neophyte would decline to purchase Google's stock.

As a side remark, we comment that although the growth rate of 35.6% is estimated for 2006 only, we find ourselves in a state of utter disbelief when we apply the principles of Graham (1985) or Whitman and Shubik (1979) to Google's balance sheet. In discordance with the "price targets" offered by financial analysts at the same finance.yahoo.com Web site, we hereby make the public prediction that Google's stock price will fall below $50 by December 31, 2010. Here again, the quantitative methods give us the ability and confidence to provide Neophyte with concrete predictions.

We are not convinced that the theory of behavioral finance offers similar features to Neophyte. Judging by Professor Belden's description of that field, it appears that Joe will certainly receive an explanation of what went wrong were he to purchase the stock and lose his capital. But that will be of little consolation to poor Joe, for he surely would have preferred valuable advice going forward over a history lesson coupled with a lifetime of regret. Having written this paragraph, we are struck by how it renews some concerns expressed by Durand (1968) about the "new finance men" having "lost virtually all contact with terra firma" (and that too from an expert in geophysics!).

Even today, Durand's brilliant article continues to provide its readers with a cogent procedure for quantifying the extent of irrational exuberance among the speculative public. We urge Professor Belden to join those of us who have read Durand's paper repeatedly over the years, who continue to be fascinated by it, and who have benefited from its application in the financial markets. In conclusion, we urge academics, investors, and speculators to remain *steadfast* with the St. Petersburg paradox to better understand growth stock valuation and to detect and quantify irrational exuberance.

3. WHITWORTH AND THE ST. PETERSBURG PARADOX

About Whitworth (1901), who proposed a resolution of the St. Petersburg paradox, we wrote in our article "Whitworth assumed that prudent gamblers would place at risk a fixed *percentage*, rather than a fixed *amount*, of their funds, and he developed a procedure for analyzing ventures that involve risk of ruin."

In private communications in November and December, 2004, Professor H. C. Tijms wrote very kindly that he "enjoyed" our "lucidly written" article; we are grateful to him for those gracious sentiments. Professor Tijms mentioned our remark (on p. 227) wherein we referred to Whitworth (1901) who proposed a betting strategy employing a percentage of one's bankroll rather than a fixed amount. Professor Tijms wrote that he found our remark striking because "the strategy of betting a fixed percentage of your bankroll in the situation of a favorable betting opportunity is known as the Kelly betting strategy," named after Kelly (1956). Professor Tijms noted that Kelly betting strategies are used extensively in statistics, operations research, and finance, and commented that an Internet search would return many references. (At time of writing, a search via Google using the key words "*Kelly betting finance* " located about 160,000 references.) Professor Tijms mentioned that his recent book (Tijms 2004) pays considerable attention to the Kelly strategy and its applications, and he urged us to bring to the attention of the statistics community the fact that Whitworth's proposal of proportional wagering long preceded Kelly's article.

Subsequent to our correspondence with Professor Tijms, we searched the literature to determine when Whitworth first derived his results on the paradox. In a profound article on the centenary of the first edition of *Choice and Chance*, Irwin (1967) gave an outline of Whitworth's life and work, paid tribute to Whitworth's contributions to the subject of probability theory, and reviewed the considerable influence of *Choice and Chance* on the subsequent development of probability. Irwin (pp. 150 and 160) rightly concludes that Whitworth's "genius" and "greatest contribution to the subject lay in his ability to use elementary algebraical methods to solve problems of very considerable complexity" and "with an exposition which is always of extreme clarity . . ."

To determine when Whitworth first proposed the strategy of proportional allocation of capital to risky ventures, we again turn to Irwin (1967). On p. 149, in a section entitled "Choice and Chance" Irwin stated that Whitworth's "second edition (1870) added three new appendices" including one "on The Disadvantage of Gambling. . . . The third edition (1878) was in essence just a rearrangement of the second." Because the results on the paradox appear in a chapter entitled "The Disadvantage of Gambling," we infer that Whitworth introduced the proportionate strategy in the second edition (1870) of *Choice and Chance*. Irwin also mentions (at the top of p. 149) Whitworth's 1895 sermon, "The Law of the Wager." Clearly, Whitworth long predated Kelly on the matter of proportionate betting. As Professor Tijms observed, "Whitworth precedes Kelly by almost 100 years, and that too in a time during which probability theory was still in its infancy."

In reviewing Whitworth (1901) at greater length, we find there a wealth of material on the St. Petersburg paradox. Indeed, the brief remarks in our article did not do full justice to Whitworth's remarkable results. For instance, Whitworth provides an explicit formula for the entrance fee to a St. Petersburg game, that formula being determined by the size of the player's available funds. We take this opportunity to describe Whitworth's contributions more fully.

In his Chapter XI, which is entitled "On the Disadvantage of Gambling," Whitworth considers a lottery, offering prizes worth $P_1, P_2, P_3, \ldots$, in which the probabilities of winning those prizes are $p_1, p_2, p_3, \ldots$, respectively. In Propositions LXIX–LXXI, he determines "what price may be paid for a ticket by a man whose available fund is n, so that by repeating his operation an average number of gains may balance an average number of losses." Assuming that the price, X, to be paid is small in comparison to n, Whitworth applies elementary mathematical considerations based on the concept of expected value to deduce that X satisfies the equation

$$\left(1 + \frac{P_1}{n} - \frac{X}{n}\right)^{p_1} \left(1 + \frac{P_2}{n} - \frac{X}{n}\right)^{p_2} \left(1 + \frac{P_3}{n} - \frac{X}{n}\right)^{p_3} \cdots = 1,$$

and deduces the approximate solution,

$$X \simeq \frac{\left(1 + \frac{P_1}{n}\right)^{p_1} \left(1 + \frac{P_2}{n}\right)^{p_2} \left(1 + \frac{P_3}{n}\right)^{p_3} \cdots - 1}{\frac{p_1}{n+P_1} + \frac{p_2}{n+P_2} + \frac{p_3}{n+P_3} + \cdots}. \quad (1)$$

It is this latter formula which Whitworth used as the basis for his analysis of the St. Petersburg paradox.

In his formulation of the paradox, Whitworth supposed that, on repeated tosses of a fair coin, the player receives 2^{j-1} "florins" if the first head is observed on the jth toss. This game may be viewed as a lottery in which prizes valued $P_1 = 1, P_2 = 2, P_3 = 4, \ldots$ are offered with corresponding probabilities $p_1 = 1/2, p_2 = 1/4, p_3 = 1/8, \ldots$. Whitworth deduces from (2) that, to enter the St. Petersburg game, the player should

pay an entrance fee of

$$\frac{\frac{4}{n}\left(1+n\right)^{1/2}\left(1+\frac{n}{2}\right)^{1/4}\left(1+\frac{n}{4}\right)^{1/8}\left(1+\frac{n}{8}\right)^{1/16}\cdots-2}{\frac{1}{n+1}+\frac{1}{2(n+2)}+\frac{1}{4(n+4)}+\frac{1}{8(n+8)}+\cdots}\quad(2)$$

florins. Whitworth also simplifies this formula for the case in which n is a power of 2.

Whitworth clearly saw his formula (2) as a satisfactory resolution of the St. Petersburg paradox. Indeed, on pp. 246–247 he commented "We have not assigned any new value to the mathematical expectation [in the St. Petersburg paradox]; … We have simply determined the terms at which a man may purchase a contingent prospect of advantage, so that by repeating the operation—*each time on a scale proportionate to his funds at that time*—he may be left neither richer nor poorer when each issue of the venture shall have occurred its own average number of times."

We note that the formula of Kelly (1956) for determining the size of a bet follows from Whitworth's formulas. Indeed, by a straightforward application of the binomial theorem to the formula at the bottom of p. 247 of Whitworth (1901), we find that the approximation derived from Whitworth's formula is precisely the same as Kelly's approximation for the entrance fee to be paid to a similar game. Specifically, for a lottery with one prize P, probability of success p, and where n is large compared to P, it follows from Whitworth's formula that

$$X\simeq n\left[\left(1+\frac{P}{n}\right)^{p}-1\right]\simeq pP-\frac{p(1-p)P^2}{2n}.$$

In Kelly's notation, $pP\equiv E$, $n\equiv B$, and $p(1-p)P^2\equiv V$; therefore, Whitworth's formula reduces to $X\simeq E-(V/2B)$, which is the same as Kelly's approximation.

We cannot resist the temptation to note that, had the 1990s high-tech stock speculators performed the *elementary* mathematical calculations provided by Whitworth's formulas (1)–(2), the terrible losses of that period might have been abated. That, of course, would have required them to acknowledge—and understand—the connection between the valuation of growth stocks and the St. Petersburg paradox.

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