Chapter 40

Maximizing Capital Growth With Black Swan Protection

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1. Introduction

Consider an investor who allocates a fixed fraction of his wealth to a stock index at the start of each period and the remainder to U.S. Treasury bills. We ask whether the investor can be better off buying calls on the index instead, in order to limit his one-period downside risk (in Nassim Taleb's evocative terminology, bad "Black Swans")?

First, we assume that stock price changes follow a stationary lognormal distribution — the world of Black–Scholes and Long Term Capital Management. Taleb calls this world where Gaussian statistics prevail Mediocristan. Then we explore the world of "Extremistan," the distribution of price changes with much fatter tails than the lognormal.

2. Assumptions and Formulas for Mediocristan

Portfolios are limited to either T-bills plus a stock index or T-bills plus call options on this stock index. We assume no transactions costs for simplicity. Call options are priced using the Black–Scholes model and are European, i.e., only exercisable at expiration. The index pays no dividends, again for simplicity.

The portfolio is revised annually. The one period arithmetic mean and variance, μ and σ^2, are related to the one period mean and variance of log X, m and s^2, as follows:

$$E(X) = \mu$$

$$Var(X) = \sigma^2 \quad \text{(arithmetic)}$$

$$\begin{cases} \mu = \exp(m + s^2/2) \\ \sigma^2 = \{\exp(s^2) - 1\}\exp(2m + s^2) \end{cases} \tag{1}$$

We are typically given the arithmetic mean and standard deviation but we'll be using the mean and standard deviation of log X, m and s because they give a much

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simpler form to the Black–Scholes formula. Solving (1),

$$m + s^2/2 = \log \mu$$

$$\sigma^2 = \{\exp(s^2) - 1\}\mu^2$$

$$\sigma^2/\mu^2 + 1 = \exp(s^2)$$

$$\begin{cases} s^2 = \log(1 + \sigma^2/\mu^2) \\ m = \log \mu - \dfrac{1}{2}\log(1 + \sigma^2/\mu^2) \end{cases} \tag{2}$$

For example, if the annual parameters are

$$\mu = 1.10, \quad \sigma^2 = 0.04, \quad \text{then } \sigma^2/\mu^2 = .033 \quad \text{and}$$

$$s^2 = 0.0325 \quad s = 0.1803 \quad m = 0.0790$$

Conversely, if $m = 0.10$ and $s = 0.20$ then $\log \mu = 0.12$, $\mu = 1.1275$, $\sigma^2 = 0.0519$, $\sigma = 0.2278$. We use these values, which are similar to those for indexes of stocks of large U.S. companies for the period 1926–2010 (Bertocchi *et al.*, 2010).

For periods other than a year, replace m by mt and s^2 by v^2t, where v is the traditional volatility of the stock as used in the BS formula.

It is convenient now to specify m and s and compute μ and σ but in Part 2 we plan to use distributions other than the lognormal, where m and s may not be explicit, so then we'll use the corresponding values of μ and σ.

From the put-call parity theorem, there should be an equivalence between portfolios which consist of long puts plus long equity, and portfolios which are long the matching calls and long T-bills.

The put-call parity theorem for European options, with no dividends paid by the underlying security and no transactions costs, is:

$$S = C - P + Ke^{-rT} \tag{3}$$

where K is the strike price of both put P and call C, r is the riskless rate, T is the common time to expiration for P and C, and S is the stock (or equity index) price.

The well-known argument is this: at expiration, both sides have gained exactly S so their initial prices must be the same (to prevent arbitrage).

For computer code to compute P or C, see Haug, *The Complete Guide to Option Pricing Formulas*, page 3 and Appendix A.

Rewriting (3) as

$$S + P = C + Ke^{-rT} \tag{4}$$

shows that equity plus a put is equivalent to a call plus a T-bill.

To insure against major downside losses, we might consider two types of portfolios, corresponding to the two sides of equation (4).

Type I: A (weighted) combination of long S, long P and long or short T-bills.
Type II: A (weighted) combination of C long and T-bills long or short.

As they are equivalent, we look at the simpler Type II.

Portfolio weights add to 1. Weights are also limited so as to prevent total loss. For a portfolio consisting of call options and T-bills, this means the portfolio is long in both options and T-bills.

The Black–Scholes formula:

$$\begin{aligned}
C &= SN(d_1) - Ke^{-rT}N(d_2)\\
P &= -SN(-d_1) + Ke^{-rT}N(-d_2)\\
d_1 &= \frac{\ln(S/K) + (r + v^2/2)T}{v\sqrt{T}}\\
d_2 &= \frac{\ln(S/K) + (r - v^2/2)T}{v\sqrt{T}}\\
&= d_1 - v\sqrt{T}
\end{aligned}$$

K striKe
r riskless (continuously compounded) rate
v volatility
m (geometric) growth rate of stock
T time until option expiration
N is the standard $N(0, 1)$ normal distribution.

We assume the initial capital $W = 1$ and the initial equity index price $S = 1$.

Next we compute C and invest an amount f in C.

Example: $C = 0.2$, we invest $f = .04$ in C so we buy 2 calls. In general, f buys f/C calls.

Note: $f \geq 0$ and $f < 1$ to avoid total loss.

We invest the remainder, $1 - f$ in T-bills, which grows to $(1 - f)e^{rT}$.

The future values at time T of the call, C^*, the stock, S^*, and the wealth, W^*, are random variables marked with asterisks.

One call pays $\max(S^* - K, 0) = C^*$ at time T.

Investing f in calls buys f/C calls, which pay

$$\frac{f}{C}C^* = \frac{f}{C}\max(S^* - K, 0)$$

The portfolio payoff at time T is W^*. To find the Sharpe ratio we need $E(W^*)$ and $\sigma(W^*) = \sqrt{V(W^*)}$ where $V(W^*)$ is the variance of W^*.

$$E(C^*) = E[\max(S^* - K, 0)] = \int_{x=K}^{\infty} (x - K)q(x)dx$$

where $q(x)$ is the lognormal:

$$q(x) = \frac{1}{xv\sqrt{T}\sqrt{2\pi}} \exp\left\{\frac{-(\ln x - \ln S - mT)^2}{2v^2T}\right\}.$$

(Note: For convenience, we chose $S = 1$ so $\ln S = 0$.)

$$E[(C^*)^2] = \int_K^\infty (x-K)^2 q(x)dx$$
$$V(C^*) = E[(C^*)^2] - [E(C^*)]^2.$$

Since the portfolio payoff W^* is:

$$W^* = (1-f)e^{rT} + \frac{f}{C}\max(S^* - K, 0) \tag{5}$$

$$E(W^*) = (1-f)e^{rT} + \frac{f}{C}E(C^*)$$

and

$$V(W^*) = V\left(\frac{f}{C}E(C^*)\right) = \left(\frac{f}{C}\right)^2 V(E(C^*))$$
$$= \left(\frac{f}{C}\right)^2 \{E[(C^*)^2] - [E(C^*)]^2\}$$

The Sharpe ratio is

$$\mathit{Sharpe}(W^*) = \frac{(E(W^*/W) - 1) - (e^{rT} - 1)}{\sqrt{V(W^*/W)}} = \frac{E(W^*) - e^{rT}}{\sqrt{V(W^*)}}$$

which simplifies to

$$\frac{E(C^*) - Ce^{rT}}{\{E[(C^*)^2] - [E(C)]^2\}^{0.5}} \tag{6}$$

and is independent of f.

Since $S = 1$, if $K = 0$ so $C = S$ then

$$E(S^*) = \exp\{(m + v^{2/}2T)\}$$

$$\mathrm{Var}(S^*) = \{\exp(v^2T) - 1\}\exp[(2m + v^2)T]$$

$$\mathit{Sharpe}(S^*) = \frac{\exp\{(m + v^{2/}2)T\} - e^{rT}}{(\{\exp(v^2T) - 1\}\exp[(2m + v^2)T])^{1/2}} \tag{7}$$

should be the limiting value as $K \downarrow 0$.

Note that the arithmetic Sharpe Ratio is independent of f *for any* investment:

R_I = return on any investment I
R_O = return on riskless investment

Invest in I,

$$Sharpe(I) = \frac{E(R_I) - R_O}{\sigma(R_I)}$$

Invest f in $I, \bar{f}$ in R_O, where $\bar{f} = 1 - f$, and

$$Sharpe(f) = \frac{\{E(fR_I) + \bar{f}R_O\} - R_O}{f\sigma(R_I)} = \frac{fE(R_I) - fR_O}{f\sigma(R_I)} = Sharpe(I)$$

Note that

$$Sharpe(I) = \frac{E(R_I) - R_O}{\sigma(R_I)} = \frac{\{E(R_I) + 1\} - (R_O + 1)}{\sigma(R_I)} = \frac{E(W_I) - e^{rT}}{\sigma(W_I)} \quad (8)$$

where we use the fact that $\sigma(R_I) = \sigma(R_I + 1) = \sigma(W_I)$ since adding a constant doesn't affect the standard deviation.

3. Geometric Growth, Standard Deviation and Sharpe Ratio

Next, we wish to study how, for fixed $K, E \log W^*$ varies with f, where W^* is given by equation (5). Note that $0 \leq f < 1$ since with $f = 1$ and $K > 0, W^*$ can be 0, with positive probability and $\log 0$ is not defined $(-\infty)$. This limitation to $f < 1$ for periodic portfolio revisions is not necessary when the portfolio is *continuously* adjusted because the portfolio can be revised before total loss occurs.

From eqn. (5),

$$G(W^*) \equiv \log W^* = \log\left\{(1-f)e^{rT} + \frac{f}{C}\max(x-K, 0)\right\} \quad (9)$$

$$g(W^*) \equiv E(G(W^*)) = E\log W^* \quad (10)$$

$$\begin{aligned} g(W^*) &= \int_0^\infty (\log W^*)q(x)dx \\ &= \int_0^K \log\{(1-f)e^{rT}\}q(x)dx \\ &\quad + \int_K^\infty \log\left\{(1-f)e^{rT} + \frac{f}{C}(x-K)\right\} q(x)dx \end{aligned}$$

where we replaced $\max(x - K, 0)$ in the last expression by $x - K$ since they are equal over the range of integration, $K \leq x < \infty$.

We compute the corresponding values for stddev $(G) = \text{sqrt}(\text{Var}\, G)$ using

$$\text{Var}\, G = E(G^2) - [E(G)]^2 \quad (11)$$

Note that $E(G) \equiv g(W^*)$ has already been computed from equation (10). Similarly, we have

$$E(G^2) = \int_0^\infty (\log W^*)^2 q(x)dx \tag{12}$$

$$= \int_0^K [\log\{(1-f)e^{rT}\}]^2 q(x)dx$$

$$+ \int_K^\infty \left[\log\left\{(1-f)e^{rT} + \frac{f}{C}(x-K)\right\}\right]^2 q(x)dx$$

$$\text{Sharpe}(G) = \frac{g(W^*) - r}{\text{stddev}(G)} \tag{12s}$$

Values of $g = g(W^*), v = \text{stddev}(G)$, and Sharpe (G) were computed from equations (10), (11), (12) and (12s) using Mathematica. They are displayed in Tables 1, 2 and 3.

The tables show there are very different portfolios which have approximately the same g and $\sigma(G)$ as the index.

For example,

Portfolio 1: $K = 0.9$, $f = 0.2$ has $g = 0.0986, \sigma(G) = 0.2103$, close to the index at $K = 0.0, f = 1.0, g = 0.1000, \sigma(G) = 0.2000$.

Also interesting is

Portfolio 2: $K = 0.8, f = 0.2$ which has $g = 0.0918, \sigma(G) = 0.1580$, giving up some growth for a risk reduction of about 1/5.

The Efficient Frontier

To compute values of g, σ and Sharpe for portfolios combining the index and T-bills, with $K = 0, f = 0.1$ to 0.9, we modify equations (9), (10) and (12) by replacing $\frac{f}{c}\max(x-K, 0)$ with fx, yielding

$$G(W^*) \equiv \log W^* = \log\{(1-f)e^{rT} + fx\} \tag{9a}$$

$$g(W^*) = \int_0^\infty \log\{(1-f)e^{rT} + fx\}q(x)dx \tag{10a}$$

$$E(G^2) = \int_0^\infty [\log\{(1-f)^{erT} + fx\}]^2 q(x)dx \tag{12a}$$

where the first integral drops out for (10a) and (12a) because $K = 0$.

The column $f = 1$ is the special case where no money is invested in T-bills. However, we can't put it all in call options with K fixed and greater than zero because that gives a positive probability of total loss, which is forbidden by the policy $\max E\log(W^*)$.

The sole admissible $f = 1$ investment is 100% in the index so we have, for the $K = 0$ rows only, $g = 0.10$, $v = 0.20$, Sharpe $= 0.25$, as shown.

Table 1: Geometric Growth, Log Normal Model.
Mean Yearly Geometric Growth: m = .1, Yearly Volatility: v = .2, Riskless Rate: r = .05, Option Life: T = 1 year
Strike Price: K, Fraction of Capital invested: f

K f	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.0570	0.0635	0.0695	0.0751	0.0803	0.0850	0.0894	0.0933	0.0968	0.1000
0.1	0.0577	0.0648	0.0713	0.0773	0.0828	0.0878	0.0923	0.0963	0.0998	
0.2	0.0586	0.0664	0.0735	0.0800	0.0858	0.0910	0.0956	0.0996	0.1030	
0.3	0.0596	0.0684	0.0762	0.0832	0.0894	0.0948	0.0994	0.1032	0.1063	
0.4	0.0610	0.0708	0.0795	0.0871	0.0936	0.0990	0.1035	0.1069	0.1093	
0.5	0.0629	0.0741	0.0837	0.0919	0.0986	0.1039	0.1077	0.1099	0.1105	
0.6	0.0655	0.0785	0.0893	0.0978	0.1043	0.1086	0.1105	0.1098	0.1057	
0.7	0.0692	0.0845	0.0962	0.1045	0.1094	0.1106	0.1076	0.0990	0.0813	
0.8	0.0742	0.0918	0.1034	0.1095	0.1097	0.1032	0.0883	0.0611	0.0097	
0.9	0.0801	0.0986	0.1073	0.1068	0.0964	0.0744	0.0368	−0.0257	−0.1421	
1	0.0856	0.1014	0.1019	0.0881	0.0591	0.0116	−0.0614	−0.1772	−0.3905	
1.1	0.0884	0.0953	0.0803	0.0458	−0.0097	−0.0910	−0.2092	−0.3903	−0.7188	
1.2	0.0858	0.0760	0.0390	−0.0219	−0.1084	−0.2272	−0.3928	−0.6400	−1.0810	
1.3	0.0753	0.0425	−0.0200	−0.1082	−0.2249	−0.3783	−0.5860	−0.8897	−1.4237	
1.4	0.0565	−0.0012	−0.0877	−0.1998	−0.3413	−0.5219	−0.7616	−1.1070	−1.7076	
1.5	0.0322	−0.0476	−0.1529	−0.2825	−0.4416	−0.6409	−0.9020	−1.2745	−1.9174	
1.6	0.0071	−0.0889	−0.2069	−0.3479	−0.5179	−0.7285	−1.0024	−1.3909	−2.0582	
1.7	−0.0145	−0.1208	−0.2464	−0.3939	−0.5701	−0.7870	−1.0679	−1.4650	−2.1455	
1.8	−0.0306	−0.1427	−0.2724	−0.4233	−0.6027	−0.8229	−1.1074	−1.5089	−2.1961	
1.9	−0.0413	−0.1564	−0.2881	−0.4408	−0.6218	−0.8436	−1.1298	−1.5334	−2.2238	
2	−0.0478	−0.1643	−0.2971	−0.4505	−0.6323	−0.8548	−1.1418	−1.5465	−2.2384	

Table 2: Standard Deviation of Geometric Growth, Log Normal Model.
m = .1, v = .2, r = .05, T = 1, Strike: K, Fraction Invested: f

K f	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.0214	0.0423	0.0627	0.0829	0.1027	0.1224	0.1419	0.1613	0.1806	0.2000
0.1	0.0236	0.0466	0.0691	0.0912	0.1131	0.1347	0.1561	0.1775	0.1990	
0.2	0.0263	0.0519	0.0769	0.1015	0.1258	0.1498	0.1737	0.1977	0.2217	
0.3	0.0298	0.0586	0.0868	0.1145	0.1418	0.1689	0.1960	0.2233	0.2508	
0.4	0.0343	0.0674	0.0996	0.1313	0.1626	0.1939	0.2253	0.2571	0.2898	
0.5	0.0403	0.0791	0.1169	0.1540	0.1910	0.2281	0.2658	0.3048	0.3459	
0.6	0.0490	0.0959	0.1415	0.1866	0.2318	0.2780	0.3261	0.3779	0.4371	
0.7	0.0620	0.1209	0.1781	0.2351	0.2932	0.3541	0.4202	0.4964	0.5963	
0.8	0.0817	0.1580	0.2319	0.3060	0.3828	0.4654	0.5590	0.6743	0.8435	
0.9	0.1105	0.2103	0.3057	0.4011	0.5006	0.6094	0.7358	0.8975	1.1494	
1	0.1502	0.2778	0.3964	0.5135	0.6349	0.7678	0.9234	1.1249	1.4453	
1.1	0.2006	0.3559	0.4944	0.6276	0.7638	0.9115	1.0834	1.3055	1.6584	
1.2	0.2584	0.4343	0.5830	0.7219	0.8611	1.0096	1.1805	1.3990	1.7431	
1.3	0.3156	0.4980	0.6440	0.7760	0.9055	1.0415	1.1957	1.3907	1.6942	
1.4	0.3605	0.5327	0.6637	0.7788	0.8895	1.0040	1.1324	1.2929	1.5400	
1.5	0.3825	0.5312	0.6396	0.7327	0.8208	0.9110	1.0110	1.1350	1.3245	
1.6	0.3772	0.4962	0.5801	0.6510	0.7173	0.7845	0.8587	0.9501	1.0890	
1.7	0.3480	0.4375	0.4992	0.5508	0.5986	0.6469	0.6999	0.7650	0.8635	
1.8	0.3032	0.3675	0.4112	0.4475	0.4810	0.5147	0.5515	0.5967	0.6649	
1.9	0.2518	0.2966	0.3268	0.3517	0.3746	0.3976	0.4228	0.4535	0.4999	
2	0.2012	0.2317	0.2521	0.2689	0.2844	0.2999	0.3168	0.3375	0.3687	

Table 3: Sharpe of Geometric Growth, Log Normal Model.
m = .1, v = .2, r = .05, T = 1, Strike: K, Fraction Invested: f

K f	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.3271	0.3194	0.3114	0.3032	0.2948	0.2863	0.2775	0.2685	0.2594	0.2500
0.1	0.3263	0.3177	0.3088	0.2997	0.2904	0.2808	0.2709	0.2608	0.2505	
0.2	0.3253	0.3157	0.3057	0.2954	0.2848	0.2739	0.2627	0.2511	0.2393	
0.3	0.3241	0.3130	0.3016	0.2897	0.2775	0.2649	0.2519	0.2385	0.2246	
0.4	0.3224	0.3096	0.2962	0.2823	0.2679	0.2530	0.2375	0.2214	0.2045	
0.5	0.3201	0.3048	0.2887	0.2719	0.2544	0.2361	0.2169	0.1966	0.1749	
0.6	0.3166	0.2976	0.2775	0.2564	0.2342	0.2107	0.1855	0.1582	0.1273	
0.7	0.3099	0.2855	0.2595	0.2320	0.2027	0.1713	0.1371	0.0987	0.0524	
0.8	0.2962	0.2644	0.2304	0.1944	0.1558	0.1143	0.0686	0.0165	−0.0478	
0.9	0.2722	0.2311	0.1876	0.1416	0.0927	0.0400	−0.0180	−0.0843	−0.1671	
1	0.2369	0.1850	0.1308	0.0741	0.0143	−0.0500	−0.1207	−0.2020	−0.3048	
1.1	0.1916	0.1273	0.0614	−0.0067	−0.0782	−0.1547	−0.2392	−0.3373	−0.4635	
1.2	0.1385	0.0600	−0.0189	−0.0996	−0.1840	−0.2745	−0.3751	−0.4932	−0.6488	
1.3	0.0800	−0.0150	−0.1086	−0.2039	−0.3036	−0.4112	−0.5319	−0.6757	−0.8699	
1.4	0.0179	−0.0962	−0.2074	−0.3207	−0.4399	−0.5696	−0.7167	−0.8950	−1.1413	
1.5	−0.0466	−0.1837	−0.3172	−0.4538	−0.5989	−0.7584	−0.9416	−1.1670	−1.4854	
1.6	−0.1137	−0.2799	−0.4429	−0.6112	−0.7917	−0.9923	−1.2256	−1.5166	−1.9360	
1.7	−0.1852	−0.3903	−0.5937	−0.8059	−1.0358	−1.2938	−1.5972	−1.9805	−2.5427	
1.8	−0.2657	−0.5243	−0.7839	−1.0578	−1.3571	−1.6961	−2.0985	−2.6126	−3.3782	
1.9	−0.3624	−0.6958	−1.0348	−1.3957	−1.7933	−2.2473	−2.7906	−3.4913	−4.5482	
2	−0.4858	−0.9250	−1.3765	−1.8612	−2.3990	−3.0170	−3.7618	−4.7304	−6.2074	

The column $f = 0$ is the special case where all the money is invested in T-bills. Since no options are purchased, K is irrelevant. The value of g is 0.05, the standard deviation is (conventionally) assumed to be zero and the Sharpe ratio is $(0-0)/0$, undefined. This column is omitted.

A look at Tables 1, 2, and 3 shows that the geometric Sharpe ratio is greatest for a given f when $K = 0$, i.e. when we buy the index rather than call options. Also the Sharpe ratio decreases, for a given f, as we increase K, i.e. raise the strike price of the call options in the portfolio.

Figure 1 plots g versus v for those points in the Table such that $g \geq 0.05$ and $v \leq 0.50$. Most of these points fall on, or very close to, a downward opening "parabola" approximating $g = .05 + .35v - .5v^2$. The part left of, and including the peak, indicates the geometric mean/standard deviation efficient frontier. Maximum compound growth occurs at about $g = 0.1106$, $v = 0.35(f = 0.6, K = 0.7)$. Note that the geometric efficient frontier is concave unlike the arithmetic efficient frontier, which with its constant Sharpe ratio, is a straight line. For instance, a straight line joining T-bills at $g = 0.05, v = 0$ with the index at $g = 0.10$, $v = .20$, crosses $v = 0.10$ at $g = 0.075$ whereas the efficient frontier value of $g = 0.08$ is about 0.5%/year higher

The points from the column $f = 0.1$ start out on the efficient frontier but as K increases they fall away in their own lower "parabola," which peaks at $K = 1.1$, $g = 0.88$, $v = 0.20$, Sharpe $= 0.19$. The points from $f = 0.2$ break away in a somewhat higher curve, peaking at $K = 1.0$, $g = 0.10$, $v = 0.28$, Sharpe $= 0.19$. The $f = 0.3$ points continue the pattern. For $f \geq 0.4$ the points remain very close to the efficient frontier over the range of values in the plot. The shaded zone in Table 1 indicates the points in the Figure which appear to fall on the efficient frontier.

From both Figure 1 and the Tables, it appears that we can produce any point on the geometric efficient frontier between T-bills ($v = 0.00, g = 0.05$) and the index ($v = 0.20, g = 0.10$) using a mix of only the index and treasury bills. Thus, for this part of the efficient frontier, using the metric of the geometric Sharpe ratio, call options on the index in the portfolio appear to have no advantage over using the index directly. However, the part of the efficient frontier between $v = 0.20, g = 0.10$ and $v = .35, g = 0.11$ comes only from call options plus T-bills. The (f, K) pairs which appear here, shaded in Table 2, are a subset of the shaded points in Table 1.

Though using options for "Black Swan insurance" seems to confer no long run growth benefits, intuition suggests we look at short and intermediate payoff structures.

4. Simulations

Because options protect against large losses, one wonders whether there are characteristics, such as less severe maximum drawdowns, that would make portfolios 1 or 2, for instance, "better" than the index. To explore this, we compare the maximum

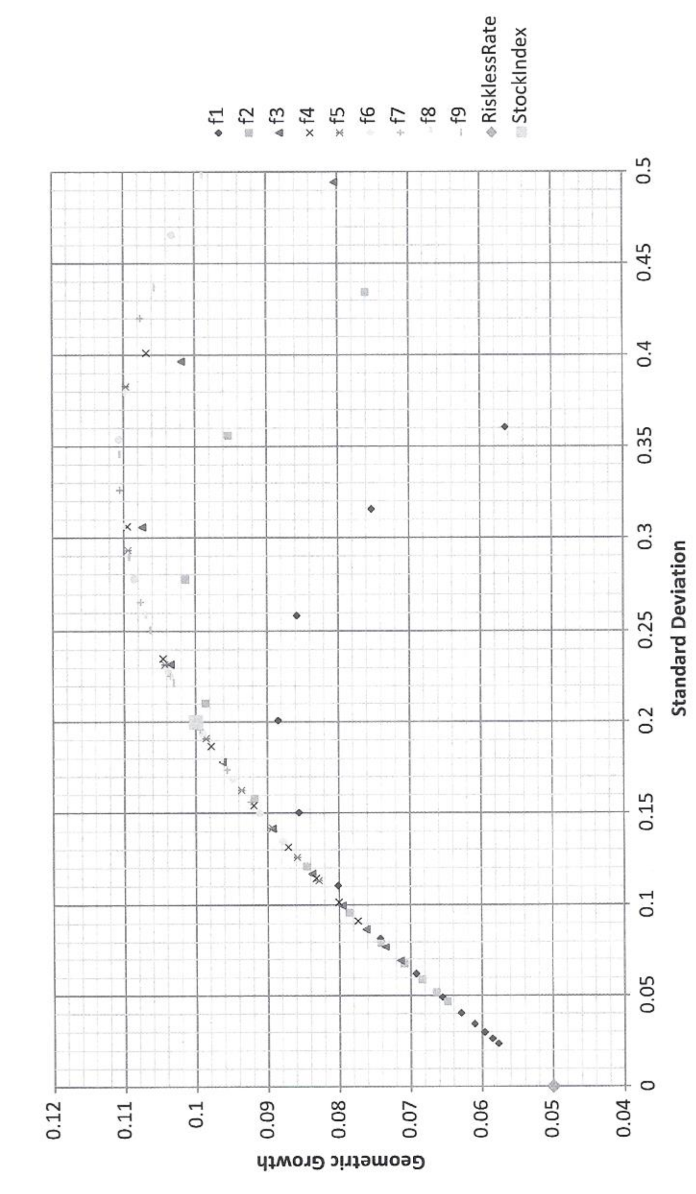

Figure 1: Geometric growth versus standard deviation for values in the Tables where $g >= .05$ and $v <= .5$.

drawdown cumulative distribution functions for some of the option portfolios with some of the index portfolios.

We selected maximum drawdown as a metric because a big drawdown in the short run often causes investors to abandon superior long term strategies. We use Monte Carlo simulations to construct the maximum drawdown cumulative distribution functions.

We simulated histograms of the probability density function and the cumulative distribution function for 100,000 random draws for each of $1, 2, 4, \ldots, 128$ periods. Let $S_1, S_2, \ldots$ be simulation results for the *index* for periods $1, 2, \ldots$. To calculate the S_i we use the random variable

$$S^* = \exp\{mT + v\sqrt{T}N^*(0,1)\} \tag{13}$$

where $N^*(0,1)$ is normal with mean 0 and variance 1.

For $T = 1$(year): $S^* = \exp\{m + vN^*(0,1)\} = \exp\{N^*(m, v^2)\}$ where $N^*(m, v^2)$ is normal with mean m and variance v^2.

If $m = 0.10, v = 0.20$, then

$$S^* = \exp\{N^*(0.10, (0.20)^2)\}.$$

If $N_1, N_2, \ldots$ are random values of $N^*(0.10, 0.20^2)$ then random values of S^* are

$$S_1, S_2, \ldots \quad \text{where } S_i = \exp\{N_i\}.$$

To simulate the index, from eqn. (15), setting $f = 1, K = 0$, and noting $C = S = 1$ for the index, we get $W_i = S_i$. To simulate portfolios having calls and T-bills, we calculate the payoff random values for the call C^* as

$$C_i = \max(S_i - K, 0). \tag{14}$$

Then from eqn. (5), we get the random values of W_i as

$$W_i = (1-f)e^{rT} + \frac{f}{C}\max(S_i - K, 0). \tag{15}$$

Note that C is the original Black-Scholes Call value at the start of the period.

As a check, it should be true that

$$\frac{1}{n}\sum_{i=1}^{n} \log W_i \to g(W^*) \tag{16}$$

which is the g we previously computed.

If W_M is the wealth after M simulated periods,

$$W_M = \prod_{i=1}^{M} W_i$$

and this, repeated many times, will give us a distribution for the maximum percentage drawdown.

After calculating Tables 1, 2 and 3, and plotting results in Figure 1, we compared several option strategies with index strategies having approximately the same g and v. The option strategy $K = 0.6333, f = 0.4$ has about the same g and v as

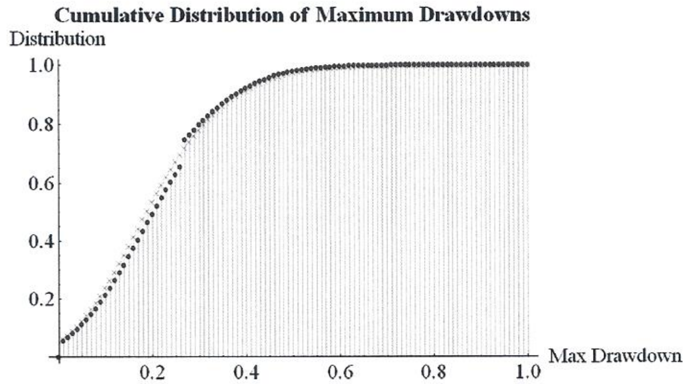

Figure 2: Comparison of the cumulative distribution of maximum drawdowns for the option strategy K = .75 and f = .3(•) and the index (×) for T = 8.

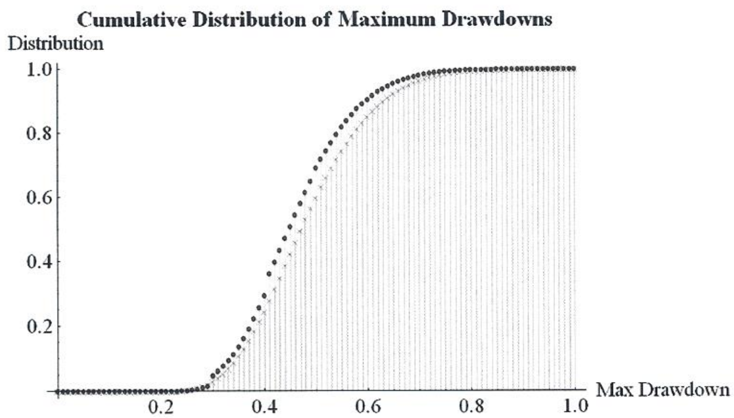

Figure 3: Comparison of the Drawdown for the option strategy K = .9 and f = .2(•) and the index (×) for $T = 100$.

the index and the distribution of maximum drawdown (DMD) was virtually the same for both strategies for all the times $T = 1, 2, 4, \ldots, 128$. The combination $K = 0.75, f = 0.3$ also has g and v close to the index, but here the one-period limited loss of the option strategy visibly affects the distribution. The DMD for the option strategy is initially lower (more small MDs), then "jumps" above the DMD for the index strategy (fewer large drawdowns). Figure 2 shows the case for $T = 8$. As T increases, the cross-over point drops towards 0 and the option strategy MDD curve moves slightly above the index MDD curve for all but small DDs.

When we reduce f further and compare the option strategy $K = 0.9, f = 0.2$ with the index when $T = 100$, the MDD curve clearly dominates that for the index over more than 95% of their range. This is despite the fact that the index has slightly better g, v and Sharpe. (See Figure 3.)

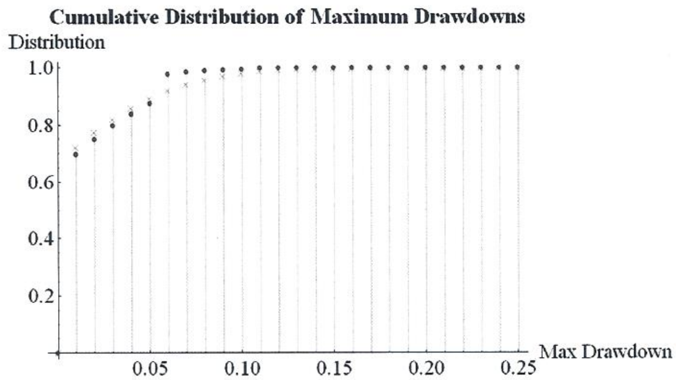

Figure 4: Option strategy f = .1, K = .8(•) versus index strategy f = .4 and K = 0 (×) for T = 2.

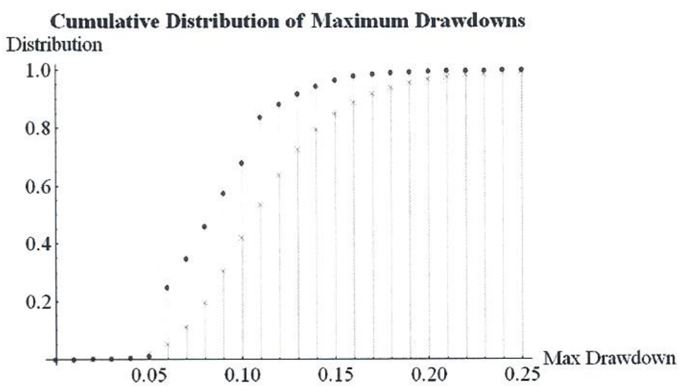

Figure 5: Option strategy f = .1, K = .8(•) versus index strategy f = .4 and K = 0 (×) for T = 64.

The effect gets even stronger as we match the option strategy $f = 0.1, K = 0.8$ with the index strategy $f = 0.4, K = 0$. Figure 4 shows the result for $T = 2$ and Figure 5 is the case $T = 64$. (Note: The horizontal axis scales are different to bring out detail.)

The possible impact of options on reducing maximum drawdown is dramatically illustrated in Figures 6 and 7. Here we compare the option portfolio $f = 0.1$, $K = 1.1$ with two index portfolios on the efficient frontier. The first index portfolio $f = 0.7$, $K = 0$, has about the same $g = 0.0894$, versus 0.0884, but a much smaller v of 0.1419 versus 0.2006 for the option portfolio. Even though the index portfolio has much less risk (measured by v), and the option portfolio is well within the efficient frontier, the DMD curves are about the same, with perhaps only a small edge for the index portfolio.

This generally holds over the entire range of $T = 1, 2, 4, \ldots, 128$. What if we match the risk v of this option portfolio with an index portfolio? Choosing $f = 1.0$,

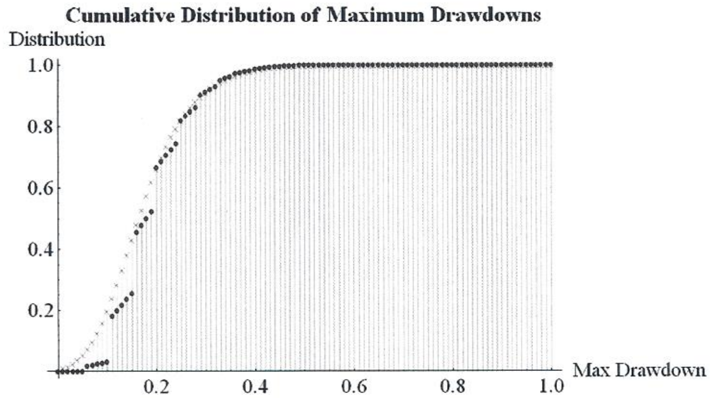

Figure 6: Option strategy f = .1, K = 1.1(•) versus index strategy f = .7, K = 0 (×) for T = 16. Riskless rate r = 5%.

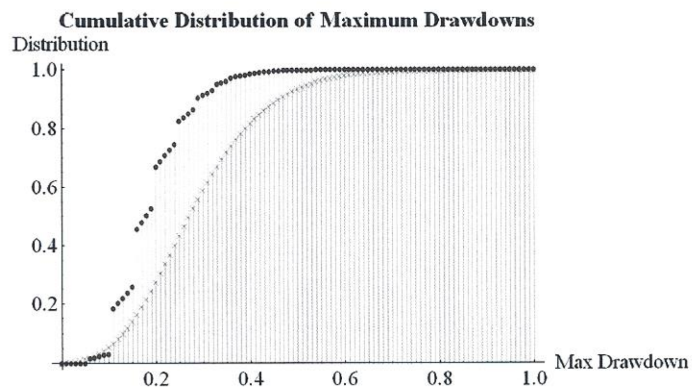

Figure 7: Option strategy f = .1, K = 1.1(•) versus index strategy f = 1.0, K = 0 (×) for T = 16. Riskless rate r = 5%.

$K = 0$ for the index portfolio, we have $g = 0.1000$ versus 0.0884 and $v = 0.2000$ versus 0.2006. Even though the index has an edge of 1.16% in expected annual long term growth, the DMD for the option portfolio, as shown in Figure 7, is much better. As T increases, the cross-over point drops rapidly towards 0 and the DMD for the option portfolio increasingly dominates.

These two examples suggest several ideas. First, if we have an option portfolio inside the efficient frontier, the index portfolio which dominates it (at least the same g, at most the same v) and has the most competitive DMD for larger T, is the index portfolio of the same g (and, of course, lesser v). Secondly, the MD benefits of the option portfolios seem to have two sources: (a) maximum loss in any one period is limited to no more than $f(r \geq 0\%)$ and (b) for comparable portfolios, more of the option portfolio's return comes from the "certain" return of T-bills, further limiting maximum one-period loss. The result is (mostly) less extreme drawdowns.

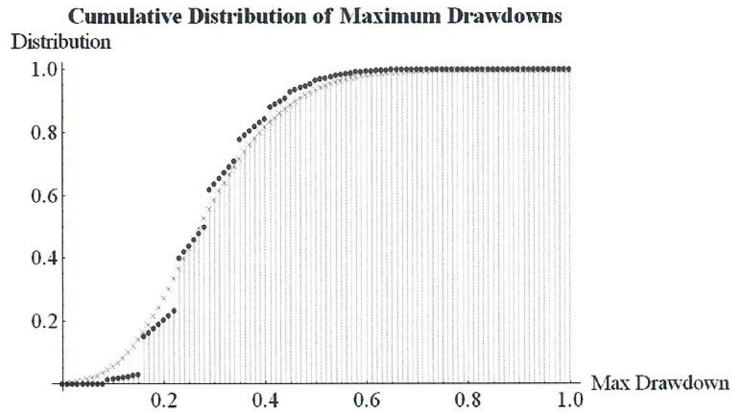

Figure 8: Option strategy f = .1, K = 1.1(•) versus index strategy f = 1.0, K = 0 (×) for T = 16. Riskless rate r = 2%.

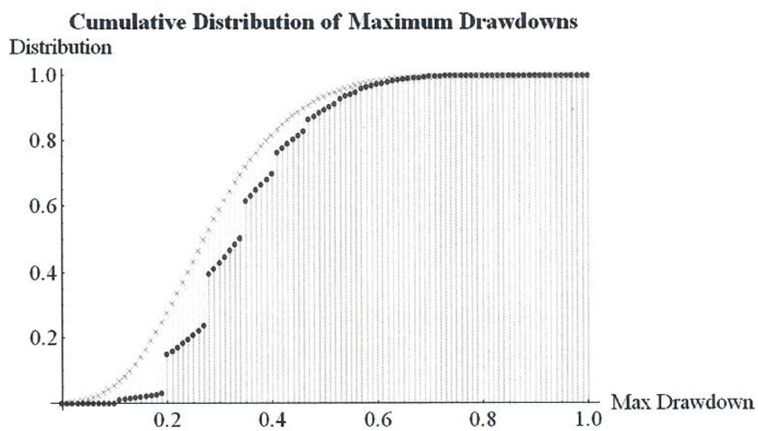

Figure 9: Option strategy f = .1, K = 1.1(•) versus index strategy f = 1.0, K = 0 (×) for T = 16. Riskless rate r = 0%.

We would expect, then, that reducing the interest rate r would shift the DMD for the option portfolio to the right. The change between Figure 7 and Figure 8 shows the effect of dropping r from 5% to 2%, for $T = 16$. The DMDs are roughly the same. Again, this holds as T increases from 16 to 128.

With $r = 0\%$, Figure 9 shows a further shift to the right of the option portfolio DMD for $T = 16$. With the option portfolio stripped of its interest income component, the superior g of the index portfolio leads to an early DMD dominance as T increases. In contrast, Figure 10 shows that increasing r to 8% gives the option portfolio overwhelmingly better MDD characteristics.

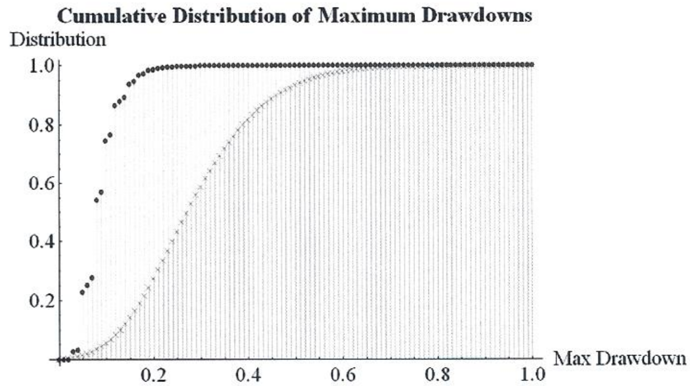

Figure 10: Option strategy f = .1, K = 1.1(•) versus index strategy f = 1.0, K = 0 (×) for T = 16. Riskless rate r = 8%.

Conclusions

In the Black-Scholes world of Mediocristan, call option portfolios which are on the geometric efficient frontier can reduce maximum drawdown risk compared to equivalent index strategies (Figures 4 and 5). The effect is stronger with a lower fraction f in the options portfolio (this reduces the achievable portion of the efficient frontier), a higher strike price K for the options (more leverage), and a higher riskless rate.

Call option portfolios which are not on the efficient frontier may substantially reduce multiperiod maximum drawdown risk at the cost of reduced long term growth or increased long term risk.

The possible advantages or tradeoffs in using call options on the index instead of the index to reduce multiperiod maximum drawdowns depend in a complex way on many factors. These include distributional assumptions e.g. (lognormal) for index returns and parameter values such as m, s and r. An important practical question is the price (compared to "fair" price) and availability of index call options.

5. Extremistan

Here we explore the world of Extremistan by using a distribution with fatter tails than the lognormal. As before, portfolios are limited to either T-bills plus a stock index or T-bills plus call options on this stock index and we assume no transactions costs. After considering various proposed fat-tailed distributions, we chose to make up our own, a t-distribution with four degrees of freedom, truncated at zero (i.e. set equal to zero for negative values of the argument). The arithmetic mean $\mu = 1.12750$ and variance $\sigma^2 = 0.05188$ match those we used for Mediocristan. The resulting density function is

$$p(x) = 72.416(4 + 36.3275(-1.12562 + x)^2)^{-5/2} \quad \text{for } x > 0 \tag{17}$$

and $p(x) = 0$ for $x < 0$

Again, the index pays no dividends and the portfolio is revised annually. Calls are European, i.e. only exercisable at expiration. Recall that the Black-Scholes formula for European calls results if we take the lognormal distribution and replace the expected growth rate by the riskless rate. If this mean-growth-rate-shifted lognormal distribution is $w(x)$ then

$$C = e^{-rt} \int_{x=K}^{\infty} (x - K) w(x) dx \tag{18}$$

However, because we have only specified a terminal distribution of index prices, and not one resulting from a process with known transition probabilities, we don't have a "no arbitrage" model for call option pricing.[1] Proceeding by analogy with equation (18), and choosing $T = 1$ (year), we priced the options in a risk-neutral setting using the fat-tailed density function of equation (17).

Let $u(x)$ equal $p(x)$ after shifting its mean so that its expected growth rate equals the riskless rate, i.e.

$$\int_{-\infty}^{\infty} u(x)dx = \int_{A}^{\infty} u(x)dx = e^{r}$$

Note that when $p(x)$ is mean-shifted to get $u(x)$, the point where $u(x)$ becomes non-zero is at some number $A < 0$, whereas for $p(x)$ it was at 0.

Now calculate C from

$$C = e^{-r} \int_{x=K}^{\infty} (x - K) u(x) dx \tag{19}$$

Note that $K > 0 > A$ so we don't ever include the values of x where $u(x) = 0$.

We obtained the density function $u(x)$ by shifting the mean of $p(x)$ to $\tilde{\mu} = \exp(.05) = 1.05127$, the one year wealth relative at our assumed riskless compound growth rate of 5%. The difference between the mean of u and $\tilde{\mu}$ is $1.12749 - 1.05127 = 0.07622$ so we change -1.12562 in (17) to $-1.2562 - 0.07622 = -1.04940$ to get $u(x)$ in equation (20).

$$u(x) = 72.416(4 + 36.3275(-1.04940 + x)^2)^{-5/2} \tag{20}$$

for $x \geq -0.07622$ and

$$u(x) = 0 \quad \text{for } x < -0.07622.$$

Alternatively, we could have truncated the risk neutral t-distribution at zero, for $\tilde{\mu}$ and σ, finding it by the same process we used to find $p(x)$.

Table 4 compares the call prices we used for Extremistan with those for Mediocristan. The differences are small in magnitude across the entire range.

However, the truncated t distribution leads to much greater maximum draw-downs than in the lognormal case as Figures 11 and 12 show. Figure 11 shows the

[1] See Ekstrom, et al., *Quantitative Finance Vol. II*, No. 8, August 2011, page 1125, for a discussion of when we have no-arbitrage pricing or risk-neutral pricing models. Also see J. Huston McCulloch, "The Risk-Neutral Measure and Option Pricing Under Log-Stable Uncertainty," http://www.econ.ohio-state.edu/jhm/papers/rnm.pdf.

Table 4: Call Prices Compared for Lognormal and Truncated Student t.

K	Lognormal	Student t
.2	.809	.809
.4	.619	.620
.6	.429	.433
.8	.245	.253
1.0	.104	.105
1.2	.0325	.0299
1.4	.00785	.00861
1.6	.00158	.00315
1.8	.000286	.00142
2.0	.0000479	.000748

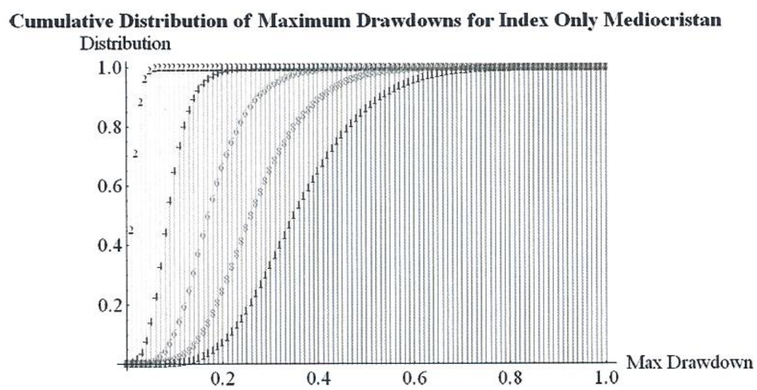

Figure 11:

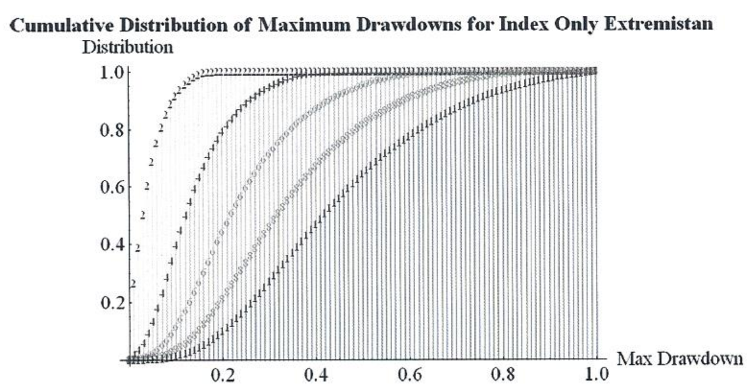

Figure 12:

simulated cumulative distribution of the maximum drawdown for the lognormal. The five curves represent fractions of $0.2, 0.4, 0.6, 0.8$ and 1.0 in the index. As the fraction increases, the curves of course move progressively to the right. Figure 12 displays the same graphs for the truncated t distribution.

6. Geometric Growth, Standard Deviation and Sharpe Ratio

Proceeding as before we compute g, σ and the Sharpe ratio from equations (9)–(12), (9a)–(12a), and (12s), replacing the lognormal $g(x)$ by the truncated Student t $u(x)$ throughout. The results, in Figure 13 and Tables 5–7, correspond to those for Mediocristan, Figure 1 and Tables 1–3.

Figure 13 shows that for $\sigma \leq 0.10$, $g \leq 0.08$, the option and index portfolios lie close to a common curve. To prefer one over the other requires a different metric. In the range $0.10 < \sigma < 0.24$, the option portfolios have a greater g for a given σ, or alternatively, less σ for a given g, with the exception of $f = 0.1$. Above $\sigma = 0.24$, the option portfolios have an efficient frontier peak at $f = 0.4$, $K = 0.9$, with $g = 0.1168$, $\sigma = 0.3593$.

We illustrate some possible tradeoffs. It is helpful to use Figure 13 to compare the points for the pairs of portfolios in our examples.

Figure 14 shows the MDD distribution for the option portfolio $K = 1.1$, $f = 0.1$, $T = 32$ years, where $g = 0.0869$, $\sigma = 0.1926$, and $Sh = 0.1915$, versus the index portfolio $f = 1.0$, which has $g = 0.0968$, $\sigma = 0.2333$, and $Sh = 0.2005$. Although the option portfolio has an annualized growth rate that is 1% less, and is well inside the efficient frontier, the reduction in MDD is enormous — a great comfort to the portfolio manager who wants to retain his clients and his job. The index portfolio has a 50% MDD with probability about 40% while the option portfolio rarely has an MDD this large. The investor Michael Korns has followed a similar strategy for more than a decade.

Just as in the Mediocristan examples, the MDD distribution at shorter times typically favors an index portfolio for small MDD and a "comparable" option portfolio for large MDD, with the option MDD becoming more dominant as T increases.

When $K = 0.9$, $f = 0.2$, $T = 32$ the option portfolio has $g = 0.0999$, $\sigma = 0.1878$, $Sh = 0.2656$ whereas the index portfolio with $f = 1.0$ is inside the efficient frontier at $g = 0.0968$, $\sigma = 0.2333$ and $Sh = 0.2005$. The dramatic reduction in "tail risk" is illustrated in Figure 15.

The portfolio with the highest growth is $K = 0.9$, $f = 0.4$. Sitting at the peak of the efficient frontier, it yields $g = 0.1168$, $\sigma = 0.3593$, and $Sh = 0.2274$. Although g and Sh are much better than the index, the MDD graphs in Figure 16 show a "ride" so wild few investors are likely to stay with it.

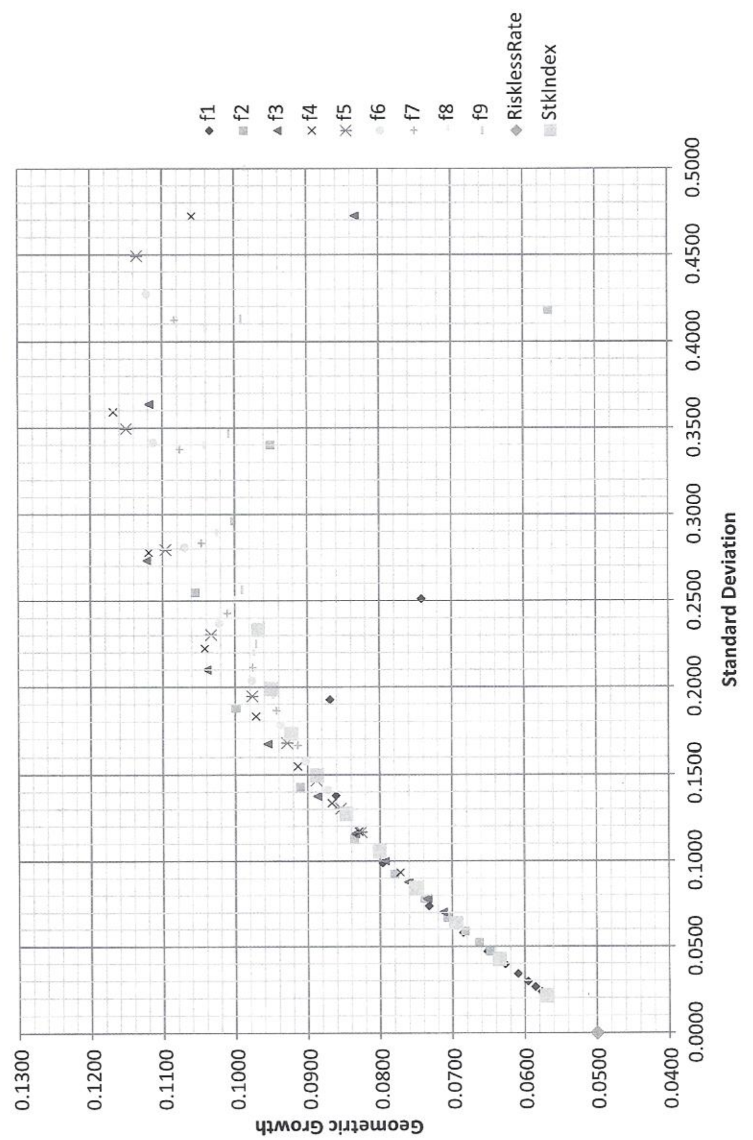

Figure 13: Geometric growth versus standard deviation for values in the Tables where $g >= .05$ and $v <= .5$. Extremistan.

Table 5: Geometric Growth for Extremistan, r = .05, T = 1 year, using truncated Student T distribution with 4 degrees of freedom, Mean of 1.12749, Variance of 0.0518805. 7/13/2011.

f–>	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
K 0.0	0.0570	0.0635	0.0695	0.0750	0.0801	0.0847	0.0887	0.0922	0.0951	0.0968
0.1	0.0577	0.0648	0.0713	0.0772	0.0826	0.0873	0.0914	0.0947	0.0971	
0.2	0.0585	0.0663	0.0734	0.0798	0.0854	0.0903	0.0943	0.0974	0.0990	
0.3	0.0596	0.0683	0.0760	0.0829	0.0888	0.0938	0.0976	0.1000	0.1004	
0.4	0.0610	0.0707	0.0793	0.0867	0.0928	0.0977	0.1011	0.1025	0.1009	
0.5	0.0627	0.0738	0.0834	0.0913	0.0976	0.1022	0.1046	0.1043	0.0991	
0.6	0.0652	0.0780	0.0886	0.0971	0.1033	0.1070	0.1076	0.1040	0.0925	
0.7	0.0685	0.0836	0.0954	0.1042	0.1096	0.1113	0.1083	0.0985	0.0756	
0.8	0.0733	0.0910	0.1039	0.1120	0.1150	0.1122	0.1019	0.0802	0.0353	
0.9	0.0796	0.0999	0.1122	0.1168	0.1135	0.1007	0.0753	0.0301	−0.0580	
1	0.0861	0.1055	0.1118	0.1059	0.0871	0.0529	−0.0026	−0.0930	−0.2623	
1.1	0.0869	0.0950	0.0832	0.0534	0.0043	−0.0685	−0.1751	−0.3391	−0.6371	
1.2	0.0741	0.0565	0.0135	−0.0522	−0.1429	−0.2655	−0.4348	−0.6857	−1.1309	
1.3	0.0493	−0.0003	−0.0763	−0.1768	−0.3052	−0.4707	−0.6920	−1.0126	−1.5723	
1.4	0.0219	−0.0540	−0.1543	−0.2786	−0.4317	−0.6243	−0.8774	−1.2394	−1.8651	
1.5	−0.0008	−0.0940	−0.2089	−0.3467	−0.5134	−0.7204	−0.9901	−1.3732	−2.0320	
1.6	−0.0174	−0.1207	−0.2436	−0.3887	−0.5624	−0.7767	−1.0547	−1.4481	−2.1229	
1.7	−0.0289	−0.1379	−0.2652	−0.4140	−0.5914	−0.8095	−1.0917	−1.4903	−2.1731	
1.8	−0.0366	−0.1489	−0.2786	−0.4296	−0.6089	−0.8290	−1.1134	−1.5148	−2.2018	
1.9	−0.0418	−0.1561	−0.2872	−0.4393	−0.6197	−0.8410	−1.1266	−1.5295	−2.2188	
2	−0.0454	−0.1608	−0.2928	−0.4456	−0.6267	−0.8486	−1.1349	−1.5387	−2.2293	

Table 6: Std. Deviation for Extremistan, r = .05, T = 1 year, using truncated Student T distribution with 4 degrees of freedom, Mean of 1.12749, Variance of 0.0518805. 7/14/2011.

f–>	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
K 0.0	0.0214	0.0425	0.0635	0.0845	0.1056	0.1272	0.1496	0.1731	0.1989	0.2333
0.1	0.0236	0.0469	0.0700	0.0932	0.1168	0.1409	0.1662	0.1935	0.2249	
0.2	0.0263	0.0522	0.0779	0.1038	0.1302	0.1575	0.1864	0.2182	0.2567	
0.3	0.0297	0.0588	0.0877	0.1169	0.1467	0.1779	0.2113	0.2488	0.2962	
0.4	0.0339	0.0670	0.1000	0.1333	0.1676	0.2036	0.2427	0.2877	0.3467	
0.5	0.0395	0.0778	0.1160	0.1546	0.1945	0.2368	0.2834	0.3382	0.4129	
0.6	0.0470	0.0924	0.1374	0.1830	0.2303	0.2810	0.3377	0.4058	0.5022	
0.7	0.0577	0.1128	0.1671	0.2221	0.2796	0.3417	0.4121	0.4987	0.6258	
0.8	0.0737	0.1426	0.2100	0.2781	0.3494	0.4271	0.5166	0.6291	0.7996	
0.9	0.0988	0.1878	0.2733	0.3593	0.4494	0.5481	0.6631	0.8103	1.0400	
1	0.1379	0.2547	0.3638	0.4719	0.5845	0.7081	0.8529	1.0405	1.3384	
1.1	0.1926	0.3401	0.4723	0.6003	0.7318	0.8752	1.0427	1.2599	1.6063	
1.2	0.2511	0.4178	0.5595	0.6926	0.8267	0.9707	1.1371	1.3510	1.6897	
1.3	0.2924	0.4559	0.5874	0.7071	0.8249	0.9492	1.0909	1.2708	1.5521	
1.4	0.3071	0.4505	0.5604	0.6576	0.7515	0.8491	0.9589	1.0967	1.3098	
1.5	0.3004	0.4185	0.5057	0.5812	0.6531	0.7270	0.8093	0.9118	1.0689	
1.6	0.2815	0.3765	0.4447	0.5028	0.5576	0.6134	0.6753	0.7519	0.8687	
1.7	0.2579	0.3339	0.3874	0.4326	0.4748	0.5177	0.5650	0.6233	0.7119	
1.8	0.2337	0.2949	0.3374	0.3731	0.4063	0.4398	0.4767	0.5221	0.5908	
1.9	0.2108	0.2607	0.2950	0.3236	0.3502	0.3769	0.4063	0.4424	0.4970	
2	0.1901	0.2312	0.2593	0.2827	0.3043	0.3261	0.3500	0.3792	0.4235	

Table 7: Sharpe for Extremistan, r = .05, T = 1 year, using truncated Student T distribution with 4 degrees of freedom, Mean of 1.12749, Variance of 0.0518805. 7/14/2011.

f->	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
K 0.0	0.3265	0.3173	0.3073	0.2965	0.2850	0.2726	0.2591	0.2441	0.2266	0.2005
0.1	0.3255	0.3152	0.3040	0.2919	0.2788	0.2646	0.2489	0.2312	0.2093	
0.2	0.3247	0.3132	0.3006	0.2869	0.2721	0.2559	0.2378	0.2170	0.1908	
0.3	0.3240	0.3111	0.2969	0.2814	0.2645	0.2459	0.2252	0.2011	0.1702	
0.4	0.3235	0.3089	0.2927	0.2750	0.2556	0.2344	0.2105	0.1826	0.1467	
0.5	0.3231	0.3063	0.2877	0.2673	0.2450	0.2204	0.1928	0.1605	0.1188	
0.6	0.3225	0.3030	0.2812	0.2575	0.2315	0.2028	0.1706	0.1330	0.0847	
0.7	0.3208	0.2977	0.2720	0.2439	0.2132	0.1794	0.1415	0.0973	0.0409	
0.8	0.3154	0.2877	0.2567	0.2229	0.1861	0.1457	0.1005	0.0481	−0.0184	
0.9	0.3000	0.2656	0.2274	0.1860	0.1413	0.0925	0.0382	−0.0246	−0.1039	
1	0.2622	0.2180	0.1698	0.1184	0.0635	0.0041	−0.0617	−0.1375	−0.2333	
1.1	0.1915	0.1324	0.0703	0.0056	−0.0624	−0.1354	−0.2159	−0.3089	−0.4277	
1.2	0.0961	0.0156	−0.0653	−0.1476	−0.2334	−0.3250	−0.4264	−0.5446	−0.6989	
1.3	−0.0025	−0.1103	−0.2151	−0.3207	−0.4306	−0.5485	−0.6802	−0.8362	−1.0453	
1.4	−0.0914	−0.2308	−0.3645	−0.4996	−0.6410	−0.7941	−0.9671	−1.1757	−1.4622	
1.5	−0.1693	−0.3439	−0.5119	−0.6825	−0.8627	−1.0598	−1.2852	−1.5609	−1.9477	
1.6	−0.2396	−0.4533	−0.6603	−0.8724	−1.0983	−1.3477	−1.6358	−1.9925	−2.5015	
1.7	−0.3058	−0.5626	−0.8135	−1.0727	−1.3507	−1.6602	−2.0206	−2.4713	−3.1229	
1.8	−0.3706	−0.6744	−0.9738	−1.2854	−1.6218	−1.9986	−2.4405	−2.9973	−3.8111	
1.9	−0.4356	−0.7905	−1.1430	−1.5120	−1.9126	−2.3637	−2.8956	−3.5702	−4.5647	
2	−0.5021	−0.9118	−1.3217	−1.7530	−2.2234	−2.7554	−3.3857	−4.1893	−5.3825	

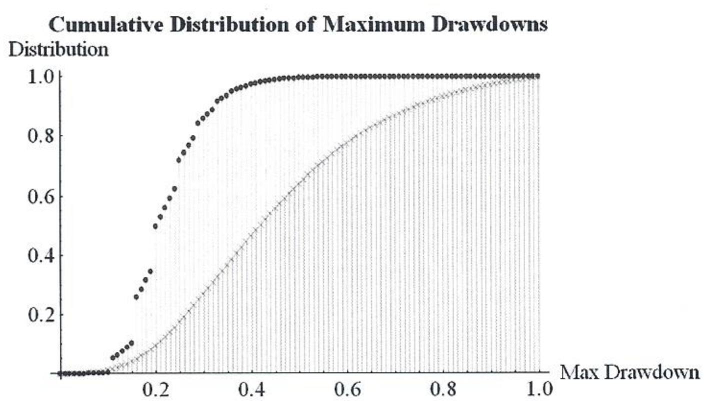

Figure 14: Comparison of the cumulative distribution of Maximum Drawdowns for Options with K = 1.1 and f = .1 for T = 32 years (•) and Stock Index (fraction = 1.0) (×).

7. Caveats and Conclusions

As there is no generally accepted fat-tailed distribution for describing equity index returns, we have made an arbitrary illustrative choice. Our truncated t-distribution is not inconsistent with the observed annual extreme returns over the last 86

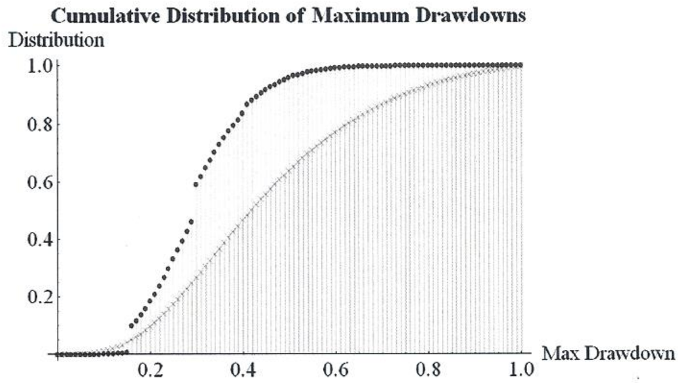

Figure 15: Comparison of the cumulative distribution of Maximum Drawdowns for Options with K = .9 and f = .2 for T = 32 years (•) and Stock Index (fraction = 1.0) (×).

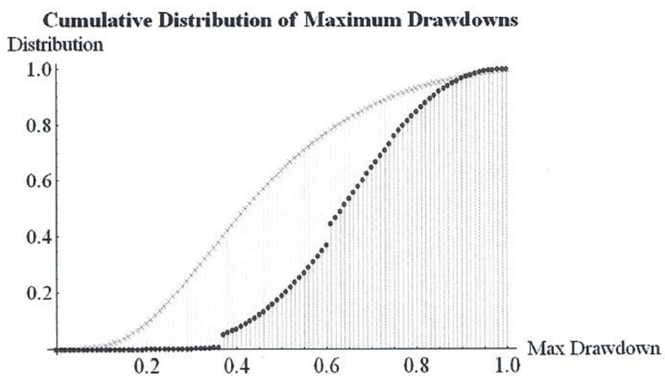

Figure 16: Comparison of the cumulative distribution of Maximum Drawdowns for Options with K = .9 and f = .4 for T = 32 years (•) and Stock Index (fraction = 1.0) (×).

calendar years. Some will argue that, even so, it is much too tame. We have also made the simplifying assumptions that options are only exercisable at expiration and that they can be priced by using the expected terminal value of the payoff, discounted at the riskless rate. Anyone who wanted to apply the methods given here would need to make their own set of choices for return distribution, option pricing model, type of option, portfolio revision period, parameters like μ, σ, and r, etc.

Nonetheless, we would expect certain qualitative features of our results to persist more generally, such as: (1) Options portfolios can attain regions of the geometric mean-variance efficient frontier beyond the reach of the index portfolios. (2) If we regard one maximum drawdown distribution as better than another if its right tail dominates (i.e. less chance of extreme MDDs), then some options portfolios are both on the efficient frontier and have better MDDs than the index portfolio which is the closest in geometric mean, standard deviation, and Sharpe ratio.

In any practical application we need to include transactions costs and the impact of taxes, either or both of which could offset the perceived advantages. One also might want to stagger option expiration dates, e.g. 1/4 per quarter, to smooth out costs, payoffs, and the need to replace — if American options are used — options which are exercised early.

Reference

Bertocchi, M., Schwartz, S., and Ziemba W., *Optimizing the Aging, Retirement and Pensions Dilemma.* Wiley, N.Y. 2010.