Rational Infinitely-Lived Asset Prices Must be Non-Stationary

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November 1, 2000

Abstract

Rational expectations must not be expected to change. Hence, a rational expectation about a future random quantity must follow a pure martingale until the uncertainty is resolved. This implies that the expectation itself could be non-stationary and, in fact, is non-stationary if the increments are iid. Most asset prices are functions of expectations about future quantities, so asset prices also could be non-stationary. This has consequences for tests based on prices rather than on returns.

Acknowledgement

The author is grateful for stimulating discussions with Stephen Ross and for helpful comments from John Cochrane, Robert Geske, Jun Liu, and Eduardo Schwartz.

I. Stochastic Processes for Rational Expectations.

If an expectation is expected to change, it cannot be rational (Samuelson [1965]). The explanation is simple: altering an expectation is costless, so if it is expected to change in the future, rationality requires a revision now.

An elementary form of this idea can be illustrated as follows: Imagine that $\tilde{V}$ is some stochastic (denoted by ~) quantity that will revealed for certain n periods from the current period t. Its current expectation is denoted $E_t(\tilde{V}_{t+n})$. Rationality requires that future expectations not be expected to change; i.e., that

$$E_t[\tilde{E}_{t+j}(\tilde{V}_{t+n})] = E_t(\tilde{V}_{t+n}), \qquad 0<j<n.$$

Since this condition holds for j=1, the general discrete stochastic process for rational expectations can be written

$$E_t(\tilde{E}_{t+1}) = E_t, \tag{1}$$

where the dependence on future $\tilde{V}$ has been notationally suppressed for convenience.

Expression (1) describes a pure martingale sequence for rational expectations. Its equivalent *ex post* form is

$$E_{t+1} = E_t + \varepsilon_{t+1} \tag{2}$$

where ε_{t+1} has mean zero and $\text{cov}(\varepsilon_{t+j},\varepsilon_t)=0$ for $j\neq0$. In the special case when ε is iid, (2) describes a random walk, which is non-stationary and thus requires special empirical methods. More generally, process (2) could be non-stationary even if the increments are not identically distributed as, for example, when their volatility is stochastic but bounded away from zero. Note that non-stationarity is a direct consequence of rationality and hence uncorrelated innovations in expectations.

II. An Application to Forward Rates.

Even the above primitive form of the rational expectations property has application to some long-cherished models in finance. My first example involves a theory of forward and spot foreign exchange rates. The unbiased forward rate hypothesis states

$$E_t(\tilde{S}_{t+n}) = F_{t,n} \tag{3}$$

where $F_{t,n}$ is the forward rate on a date t contract to deliver foreign exchange n periods in the future and S_{t+n} is the spot exchange rate on date t+n. There is no risk premium, hence no bias, according to this hypothesis. Rational expectations requires the same condition to hold in each future period before t+n, so for successive periods,

$$E_t[\tilde{E}_{t+1}(\tilde{S}_{t+n})] = E_t(\tilde{F}_{t+1,n-1}) = F_{t,n} \tag{4}$$

which shows that the sequence of forward exchange rates is a martingale.

Another example is the pure expectations hypothesis of the term structure of interest rates. It has exactly the same algebraic form and therefore implies that the sequence of forward interest rates pertaining to the same future spot loan period must follow a pure martingale.

These simple models of forward rate expectations can be augmented by the addition of risk premiums, possibly non-constant. In the case of foreign exchange, one can simply add the premium, say π_t, to (4) and write it in *ex post* form as

$$F_{t+1,n-1} = F_{t,n} + \pi_t + \varepsilon_{t+1}. \tag{5}$$

Process (5) could conceivably be stationary, but only if the *ex ante* risk premium moves inversely with the level of $F_{t,n}$ and eventually dominates the innovations represented by ε_{t+1}, the unanticipated alterations in expectations. If the variability in risk premiums is small relative to the volatility in the innovations, (5) could be non-stationary. This seems likely because forward rates are very noisy, far too noisy to be dominated by slowly varying risk premiums except near expiration when the volatility of ε vanishes.

III. Long-lived Assets.

A typical stock or bond is algebraically more complex since it emits stochastic cash flows over extended periods. For simplicity of illustration, assume first that the discount rate (or required return) is a constant for all time. The standard discounted cash flow approach gives the asset's value on date t as

$$V_t = \sum_j d^j E_t(\tilde{C}_{t+j}) \tag{6}$$

Where d is the (constant) discount factor and C_{t+j} is a cash outflow on date t+j.

Given a constant discount factor, the same cash flow formula applies in each successive period; i.e.,

$$E_t(\tilde{V}_{t+1}) = \sum_j d^j E_t[\tilde{E}_{t+1}(\tilde{C}_{t+1+j})]\,. \tag{7}$$

Applying the principle of rational expectations and simplifying, (7) can be reduced to

$$E_t(\tilde{V}_{t+1} + \tilde{C}_{t+1}) = V_t/d\,. \tag{8}$$

In structure, expression (8) is similar to the forward exchange rate sequence with a risk premium discussed above. It differs in only two respects. First, the current asset value is multiplied by 1/d, which would exceed unity in all but pathological cases. Second, there is a stochastic cash flow in the next period added to the asset's value on the left side.

To make things even easier for illustration, imagine that this asset will emit no cash flows in the immediate future. In this special case, the *ex post* process is

$$V_{t+1} = V_t/d + \varepsilon_{t+1}\,, \tag{9}$$

a submartingale in V (because $d<1$). If ε happens to be iid, process (9) is obviously non-stationary and even explosive.

The submartingale described by (8) is stationary if the price is bounded. Moreover, the convergence would be quite rapid if (a) successive values of the cash flows attenuated the process, or (b) the stochastic innovations in expectations vanished over time. For a zero-coupon bond, the stochastic innovations would in fact vanish at the maturity date.

Expressions (6) and (7) can be easily generalized to allow for maturity-dependent and stochastic discount factors. This would not, however, negate the general strong tendency for rational expectations to imply near non-stationarity unless the discount factors varied inversely with the level of asset prices.

IV. A General and Disturbing Property of Rational Expectations.

A martingale may or may not be stationary. If its increments are iid, then it becomes a non-stationary random walk. But there are other possibilities; for example, its increments could have declining volatility over time. As volatility vanishes the process ultimately anchors to some realized value, constant and trivially stationary. This would be true for forward exchange rates or interest rates. They must eventually equal a realized future spot rate. Whether or not they behave as a non-stationary process prior to that realization is an intriguing empirical question.

When a martingale sequence continues indefinitely, it can still be stationary. This is true even for a submartingale, which has positive drift. The martingale convergence theorem, Durrett [1996, p. 235], states that a submartingale sequence converges almost surely <u>to a fixed finite value</u> provided that its maximal expected positive value is bounded below $+\infty$.[1] It may be hard to visualize an economic time series, even infinitely long, which does not satisfy this last condition. It may seem unthinkable that infinity is the maximal expected value of any economic series though of course this cannot be ruled out logically in an infinitely long universe. Even if the series is bounded, convergence could take quite a while and the near-term path could be arbitrarily close to and virtually impossible to distinguish from a true non-stationary process.

[1] This result is not very intuitive. To understand it, one must remember that a martingale's increments are uncorrelated. If a series is bounded, either it displays mean reversion and therefore does not have uncorrelated increments or it must eventually cleave to a fixed finite value.

Yet it seems even more difficult to accept the astonishing economic implication of the martingale convergence theorem that a bounded series eventually settles down to a constant value. Such super-stationary is decidedly not characteristic of the typically noisy economic time series.

In general then, with rational expectations there are three possibilities: (1) the series of expectations is subject to a stationary terminal condition at a finite time; (2) the series is infinitely lived and, being bounded, converges to a constant; (3) the series is non-stationary. The first possibility applies to all assets with finite expiration dates; e.g., forward rates, derivatives, bonds. For the second condition to prevail, we must imagine some future date beyond which there is no longer any volatility in the series. This seems quite implausible for most infinitely-lived economic variables such as GNP or stock prices. Hence, we are left with the third possibility as the most likely model for their expectations. Note that martingales admit no intermediate limiting distribution; if bounded they converge to a constant while if unbounded they cannot be covariance stationary.

For infinitely lived assets such as equities, there is probably no diminution over time in the size of stochastic innovations in expectations. Moreover, to the extent that cash flows such as dividends are fractions of the expected price appreciation, it seems plausible that processes analogous to (8), even if discount rates vary through time, could be non-stationary. This suggests that tests employing functions of prices, such as size, market-to-book ratios, dividend yields, and similar constructs, should be interpreted with caution. Non-stationary processes often display seemingly bizarre behavior and can easily conjure up the specter of an “anomaly.”

But caution is not outright condemnation. Even if prices are non-stationary, dividend yields, for example, would pose few empirical difficulties so long as dividends are also non-stationary and are cointegrated with prices. Moreover, since returns are essentially first differences in prices, they should be stationary.

In conclusion, there seem to be no really appealing choices for processes to describe infinitely-lived asset price levels under rational expectations. Convergence to a fixed price has never been witnessed but non-stationarity seems equally bizarre and implausible. If neither condition is deemed acceptable, the alternative is irrational expectations. In that case, anything is possible.

References

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